Online Appendix to
Learning What to Produce:
Firm Dynamics over the Life Cycle

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This Online Appendix contains the full theoretical proofs and supporting empirical and quantitative material. The main paper includes concise proof sketches and supporting theoretical derivations.

OA.A Theory Proofs

OA.A.1 Proof of Lemma 1

Proof. Fix true expertise θ_{ip}, an observation window of length τ, and its processing path. All expectations average over potential signal histories without conditioning on product retention. From (2.3)–(2.4), the posterior mean is normal conditional on true expertise, with

$$\mathbb{E}[\hat{\theta}_{ipt} \mid \theta_{ip}] = \log \theta_{ip} + \frac{\hat{\sigma}^2_{ipt}}{\sigma_0^2}(\theta_0 - \log \theta_{ip}), \tag{OA.A.1}$$

$$\mathrm{Var}(\hat{\theta}_{ipt} \mid \theta_{ip}) = (\hat{\sigma}^2_{ipt})^2 \frac{\tau + \sum_{s=t-\tau+1}^{t} \psi_{is}}{\sigma_\varepsilon^2} = \hat{\sigma}^2_{ipt}\left(1 - \frac{\hat{\sigma}^2_{ipt}}{\sigma_0^2}\right). \tag{OA.A.2}$$

Conditional independence of the signal errors gives the variance, with the last equality following from (2.4). Holding earlier processing fixed, $\partial \hat{\sigma}^2_{ipt}/\partial \psi_{it} = -(\hat{\sigma}^2_{ipt})^2/\sigma_\varepsilon^2 < 0$, which establishes both bias comparisons. The same conclusion holds for lower cumulative processing at fixed tenure. □

OA.A.2 Destination-Product Assignment

From firm age two onward, the exclusion set contains n_i distinct associated markets: each occupied line's current product and each vacant line's latest destination. Dropping preserves the associated market until search. Distinct draws outside the exclusion set preserve this property: occupied lines keep their old market after failure, whereas vacant lines record the new draw regardless of success. At the startup's first search, only its initial product is excluded; conditional on retention, all initially vacant lines receive distinct destinations, establishing the property from age two onward.

Fix $J^s_{it} \geq 1$ searching lines. Since $J^s_{it} \leq n_i \leq \lfloor N/2 \rfloor$, the $N - n_i$ feasible destinations suffice for every combination of search decisions. Draw J^s_{it} destinations uniformly without replacement and assign them uniformly across searching lines. A feasible destination p' is selected with probability $J^s_{it}/(N - n_i)$ and, conditional on selection, assigned to a particular searching line with probability $1/J^s_{it}$. Therefore,

$$\Pr(p'_k = p' \mid \text{the feasible destination set, } k \text{ searches}) = \frac{J^s_{it}}{N - n_i}\frac{1}{J^s_{it}} = \frac{1}{N - n_i}. \tag{OA.A.3}$$

Independent search success then gives adoption probability $x/(N - n_i)$. For the startup's first search, the same argument replaces $N - n_i$ by $N - 1$.

Symmetric markets and the common-prior reset give every feasible destination the same expected

adoption value, so other lines' search decisions do not affect the focal line's expected search payoff.

OA.A.3 Threshold Properties

Consider the separable product-line problem. Maintain $0 < \beta < 1$, $0 \leq \delta < 1$, $0 < x < 1$, $\eta > 1$, $z_i > 0$, $f^o, f^s > 0$, positive finite prior and signal variances, and nonnegative signal precisions finite at each age. Aggregate prices and quantities are positive, symmetric, and stationary, while firm age and information processing follow their prescribed paths. We use the prior-reset and endogenous-exit payoff conventions in Section 2.3.7.

The argument follows the recursive verification and monotonicity methods of Stokey and Lucas Jr (1989, Theorem 9.2 and Section 9.2), with an explicit payoff bound for unbounded returns. Thresholds follow from value matching and strict crossing, using the logic of Stokey (2009, Section 6.2) in the present discrete-time setting.

OA.A.3.1 Monotonicity and Single Crossing

Lemma OA.1 (Monotonicity and Single Crossing)**.** *At fixed product tenure and firm-level states, $V_{it}^{ns}(\hat{\theta}, \tau)$ and $V_{it}^{cs}(\hat{\theta}, \tau)$ are continuous and strictly increasing in $\hat{\theta}$, and $V_{it}(\hat{\theta}, \tau)$ is continuous and weakly increasing. The differences $V_{it}^{cs}(\hat{\theta}, \tau) - V_{it}^{ds}(\emptyset)$ and $V_{it}^{ns}(\hat{\theta}, \tau) - V_{it}^{cs}(\hat{\theta}, \tau)$ are continuous and strictly increasing.*

Proof. Payoff bound. Use the posterior moment

$$H_{it}(\hat{\theta}, \tau) \equiv \mathbb{E}[\theta_{ip}^{\eta-1} \mid \mathcal{I}_{ipt}] = \exp\left\{(\eta-1)\hat{\theta} + \frac{(\eta-1)^2}{2}\hat{\sigma}_{ipt}^2\right\}.$$

The law of iterated expectations gives

$$\mathbb{E}_{it} H_{i,t+1}(\hat{\theta}', \tau+1) = H_{it}(\hat{\theta}, \tau) \tag{OA.A.4}$$

under retention. Since $\mathcal{B}_{it}^{\eta} \leq \exp\{(\eta-1)^2\sigma_\varepsilon^2/(2\eta)\} H_{it}$, there is a finite constant K, independent of firm age, such that

$$0 \leq \overline{\pi}_{it+1}(\hat{\theta}, \tau) + w_t f^o \leq K H_{it}(\hat{\theta}, \tau), \qquad |\overline{\pi}_{it+1}(\hat{\theta}, \tau)| \leq K H_{it}(\hat{\theta}, \tau) + w_t f^o. \tag{OA.A.5}$$

Search costs are bounded. A replacement resets the moment to $H^0 = \exp\{(\eta-1)\theta_0 + (\eta-1)^2\sigma_0^2/2\}$.

Under any history-dependent policy, retention contributes at most the current nonnegative moment in expectation, and each replacement adds at most H^0. Assigning moment zero to a vacancy, the expected moment after j transitions is therefore bounded by $H_{it} + jH^0$. For a finite K',

$$\begin{aligned} &\sup_{\text{feasible policies}} \mathbb{E}_{it}[\text{absolute discounted payoffs from period } t+T] \\ &\qquad \leq K' \sum_{j=T}^{\infty} \beta^j \left[1 + H_{it}(\hat{\theta}, \tau) + (j+1)H^0\right] \longrightarrow 0. \end{aligned} \tag{OA.A.6}$$

For this bound, the assigned payoff at endogenous exit is represented by the vacant-line continuation problem to which it is equal. It introduces no additional primitive payment into that continuation problem.

Consequently, zero-terminal finite-horizon values converge to the value of the infinite-horizon problem, uniformly on compact belief sets. The same bound gives value growth at most proportional to $1 + H_{it}$, uniformly in age, and bounds vacant-line values uniformly. Expected discounted terminal values vanish under every feasible policy. Finite action sets and continuous action values permit a measurable maximizing rule, so Bellman verification selects the value in this growth class.

Preservation and strict crossing. At fixed tenure and firm states, the next posterior mean equals the current mean plus a mean-independent Gaussian innovation. Using the same innovation for two current means preserves their ordering. Backward induction therefore preserves weak monotonicity of the value. Gaussian integration with the payoff bound preserves continuity. Strictly increasing current profits make the no-search branch strictly increasing; its weight $1 - x > 0$ makes the continue-and-search branch strictly increasing as well. Compact-uniform convergence and the limiting Bellman equations establish these properties at the infinite horizon. Finally, the branch equations give

$$V_{it}^{cs}(\hat{\theta}, \tau) - V_{it}^{ds}(\emptyset) = (1 - x)\left[V_{it}^{ns}(\hat{\theta}, \tau) - \beta(1 - \delta)V_{i,t+1}^{ds}(\emptyset)\right], \tag{OA.A.7}$$

$$V_{it}^{ns}(\hat{\theta}, \tau) - V_{it}^{cs}(\hat{\theta}, \tau) = w_t f^s + x\left[V_{it}^{ns}(\hat{\theta}, \tau) - V_{it}^{ns}(\hat{\theta}^0, 0)\right]. \tag{OA.A.8}$$

Both differences are continuous and strictly increasing in $\hat{\theta}$. □

OA.A.3.2 Proof of Proposition 1

Proof. The lower endpoint. Suppose (2.18) holds along the future age path. Since dropping is feasible, $V_{is} \geq V_{is}^{ds}(\emptyset)$. Subtracting vacancy value from each action, and using $V_{is}^{ds}(\emptyset) \geq \beta(1-\delta)V_{i,s+1}^{ds}(\emptyset)$, gives

$$0 \leq V_{is}(\hat{\theta}, \tau) - V_{is}^{ds}(\emptyset) \leq \beta K H_{is}(\hat{\theta}, \tau) + \beta(1-\delta)\mathbb{E}_{is}\left[V_{i,s+1}(\hat{\theta}', \tau+1) - V_{i,s+1}^{ds}(\emptyset)\right].$$

Iterate this inequality along the same product's retained-belief process. The martingale property (OA.A.4) and the value-growth bound make the discounted remainder vanish, yielding

$$0 \leq V_{is}(\hat{\theta}, \tau) - V_{is}^{ds}(\emptyset) \leq \frac{\beta K}{1-\beta(1-\delta)} H_{is}(\hat{\theta}, \tau). \tag{OA.A.9}$$

The no-search equation and (OA.A.4) now imply

$$\lim_{\hat{\theta}\to-\infty} V_{it}^{ns}(\hat{\theta}, \tau) = -\beta w_t f^o + \beta(1-\delta)V_{i,t+1}^{ds}(\emptyset). \tag{OA.A.10}$$

Thus the low-belief behavior follows from the vacancy ordering and positive operating costs; it requires no separate assumption.

Two ordered value levels. Equations (OA.A.7)–(OA.A.8) show that the cutoff equations are

$$V_{it}^{ns}(\hat{\theta}_{it}^{ds}(\tau), \tau) = \beta(1-\delta)V_{i,t+1}^{ds}(\emptyset),$$
$$V_{it}^{ns}(\hat{\theta}_{it}^{cs}(\tau), \tau) = V_{it}^{ns}(\hat{\theta}^0, 0) - \frac{w_t f^s}{x}.$$

The vacancy equation gives their difference:

$$V_{it}^{ns}(\hat{\theta}^0, 0) - \frac{w_t f^s}{x} - \beta(1-\delta)V_{i,t+1}^{ds}(\emptyset) = \frac{V_{it}^{ds}(\emptyset) - \beta(1-\delta)V_{i,t+1}^{ds}(\emptyset)}{x} > 0. \tag{OA.A.11}$$

Both levels exceed (OA.A.10). The no-search value tends to infinity as $\hat{\theta} \to \infty$, because profits diverge and continuation value is bounded below by vacancy value. Lemma OA.1 and the intermediate value theorem therefore give two unique finite, strictly ordered roots. The signs in (OA.A.7) and (OA.A.8) establish the stated action regions. The strictly increasing map $\mathcal{B}_{it}(\hat{\theta}, \tau)$ transfers existence, uniqueness, and ordering to the demand cutoffs. □

The proof only needs weak vacancy ordering at future dates and strict ordering at the current date. Whenever both pairwise roots exist, (OA.A.11) also establishes the exact equivalence

$$\hat{\theta}^{cs}_{it}(\tau) > \hat{\theta}^{ds}_{it}(\tau) \quad \Longleftrightarrow \quad V^{ds}_{it}(\emptyset) > \beta(1-\delta)V^{ds}_{i,t+1}(\emptyset).$$

With constant processing and stationary line-level primitives, vacancy value is independent of firm age, so this inequality reduces to $V^{ds}_{it}(\emptyset) > 0$. With age dependence, positivity alone does not order current and discounted future vacancy values.

A primitive sufficient condition. The following inequality implies (2.18) at every incumbent age:

$$\beta x\, \overline{\pi}_{it+1}(\hat{\theta}^0, 0) > [1 - \beta(1-\delta)(1-x)]\, w_t f^s. \tag{OA.A.12}$$

Prior expected operating profit is given explicitly by (2.10) and (2.8). At stationary aggregates it is independent of firm age. Thus this is a sufficient condition on primitives and given aggregate prices, allowing any deterministic information-processing path.

Proof of sufficiency. Since $V_{i,t+1} \geq V^{cs}_{i,t+1}$, (OA.A.7) and the no-search equation imply

$$\begin{aligned} &V^{ns}_{it}(\hat{\theta}, \tau) - \beta(1-\delta)V^{ds}_{i,t+1}(\emptyset) \\ &\quad \geq \beta\overline{\pi}_{it+1}(\hat{\theta}, \tau) + \beta(1-\delta)(1-x)\mathbb{E}_{it}\left[V^{ns}_{i,t+1}(\hat{\theta}', \tau+1) - \beta(1-\delta)V^{ds}_{i,t+2}(\emptyset)\right]. \end{aligned}$$

This inequality holds without assuming vacancy ordering. Under $\rho = 0$, the retained-product demand term satisfies $\mathbb{E}_{is}\mathcal{B}_{i,s+1}(\hat{\theta}', \tau+1) = \mathcal{B}_{is}(\hat{\theta}, \tau)$. Expected profit is convex in this demand term because $\eta > 1$. Consequently, Jensen's inequality implies that expected operating profit along the original product's learning process is at least its initial expected operating profit. Iterate the displayed inequality from the common prior. The value-growth bound makes the remainder vanish and gives

$$V^{ns}_{it}(\hat{\theta}^0, 0) - \beta(1-\delta)V^{ds}_{i,t+1}(\emptyset) \geq \frac{\beta\overline{\pi}_{it+1}(\hat{\theta}^0, 0)}{1 - \beta(1-\delta)(1-x)}.$$

The vacant-line Bellman equation then yields

$$V^{ds}_{it}(\emptyset) - \beta(1-\delta)V^{ds}_{i,t+1}(\emptyset) \geq \frac{\beta x\overline{\pi}_{it+1}(\hat{\theta}^0, 0)}{1 - \beta(1-\delta)(1-x)} - w_t f^s > 0.$$

The argument applies at every incumbent age. Uniform boundedness of vacancy values also implies

that iteration of this strictly positive difference gives $V_{it}^{ds}(\emptyset) > 0$. □

The drop margin. Substituting the no-search equation into drop value matching gives

$$\overline{\pi}_{it+1}(\hat{\theta}_{it}^{ds}(\tau), \tau) = -(1-\delta)\mathbb{E}_{it}\left[V_{i,t+1}(\hat{\theta}', \tau+1) - V_{i,t+1}^{ds}(\emptyset)\right] \leq 0, \tag{OA.A.13}$$

with the expectation evaluated at the drop cutoff. The firm retains an incumbent partly for its continuation surplus over a vacancy, so the drop threshold is weakly below the static break-even threshold. With the maintained nondegenerate Gaussian transition, the surplus is positive with positive probability and the inequality is strict.

Product values and expected demand. At the search margin a new product has a higher no-search value than the incumbent, by $w_t f^s/x$. Values also incorporate uncertainty and future learning, so this comparison need not rank their current demand terms. Equation (2.8) gives

$$\mathcal{B}_{it}(\hat{\theta}^0, 0) > \mathcal{B}_{it}^{cs}(\tau) \quad \Longleftrightarrow \quad \hat{\theta}_{it}^{cs}(\tau) < \theta_0 + \frac{\eta - 1}{2\eta}\left(\sigma_0^2 - \hat{\sigma}_{ipt}^2\right).$$

For positive tenure, $\hat{\sigma}_{ipt}^2 < \sigma_0^2$, so $\hat{\theta}_{it}^{cs}(\tau) \leq \theta_0$ is sufficient. Adopting the new product resets its beliefs and tenure, not the firm's information processing.

OA.A.4 Product-Line Separability and Additive Firm Value

Proof of Lemma 2 and Proposition 2. Fix $a_{it} \geq 2$ and the common firm-level states. Under the learning specification and prescribed processing path, each line's conditional profit and belief transition depend only on its own state. Appendix OA.A.2 establishes that every subset of searching lines is feasible and each line's expected replacement payoff is independent of other lines' states and choices.

Use the sum of the product-line values in (2.12)–(2.15) as a candidate firm value. For any tuple of decisions retaining at least one product, additive payoffs and linearity of expectation give the sum of the corresponding line-level action values. If all active products are dropped, payoff equivalence gives $n_i V_{it}^{ds}(\emptyset)$, again the sum of the line-level drop values. Maximizing each occupied line's value, with each vacant line contributing $V_{it}^{ds}(\emptyset)$, therefore maximizes firm value.

The candidate satisfies the firm's Bellman equation and exit payoffs. The single-line bound (OA.A.6), summed over n_i lines, makes discounted expected terminal values vanish and bounds omitted continuation payoffs uniformly over feasible policies. Iterating the Bellman inequality

bounds every policy's value above by the candidate sum, which the maximizing line policies attain. This verifies the sum and both results. At age one, initially vacant lines receive zero upon exit, so the startup instead uses (2.20). □

OA.A.5 Product-Line Distribution

The exact firm transition in (2.30) tracks joint portfolios. For computation, we approximate the joint distribution of belief, tenure, and vacancy states by the product of their common marginals conditional on firm age and types. This determines whether a dropped or vacant line has another active line and hence reaches search. This distributional approximation does not follow from value separability.

For computation, we extend μ_{pt} to include vacant lines. The measure $\mu_{pt}(\hat{\theta}, \tau, a)$ denotes the mass of active product lines associated with market p, while $\mu_{pt}(\emptyset, a)$ denotes the mass of vacant lines whose most recent search destination is p. Initially vacant lines of age-zero startups have not yet drawn a destination and are therefore handled directly in their first search. To simplify notation, the firm-level states (ψ, φ, m) are suppressed throughout this section; all measures, transition densities, and policy regions are understood to be conditional on their appropriate values. We retain age explicitly because the transition differs for new entrants, age-one startups, and older incumbents. Dependence on φ is shown when scope $n(\varphi)$ enters directly. We write $g(\hat{\theta}' \mid \hat{\theta}, \tau, a)$ for the transition density of the posterior mean from a line with state $(\hat{\theta}, \tau, a)$.

Approximate portfolio-continuation probability. For incumbent firms transitioning to age $a' \geq 2$, let $q^{\emptyset}_{t+1}(a')$ denote the marginal probability that a non-focal product line is vacant after product-drop decisions and before search. Define the normalized product-line distribution conditional on the suppressed firm-level states by

$$\overline{\overline{\mu}}_{qt}(\emptyset, a) \equiv \frac{\mu_{qt}(\emptyset, a)}{\mu_{qt}(\emptyset, a) + \sum_{\tau \geq 0} \int_{\mathbb{R}} \mu_{qt}(\hat{\theta}, \tau, a) d\hat{\theta}},$$

and

$$\overline{\overline{\mu}}_{qt}(\hat{\theta}, \tau, a) \equiv \frac{\mu_{qt}(\hat{\theta}, \tau, a)}{\mu_{qt}(\emptyset, a) + \sum_{\widetilde{\tau} \geq 0} \int_{\mathbb{R}} \mu_{qt}(\widetilde{\theta}, \widetilde{\tau}, a) d\widetilde{\theta}}.$$

Normalize only age and type cells with positive mass. Thus,

$$q^{\emptyset}_{t+1}(a') = \frac{1}{N-1}\sum_{q\neq p}\Bigg[\overline{\overline{\mu}}_{qt}(\emptyset, a'-1) + \sum_{\tau\geq 0}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{1}_{\mathcal{R}^{ds}_{t+1}}(\hat{\theta}', \tau+1, a') \times g(\hat{\theta}' \mid \hat{\theta}, \tau, a'-1)\overline{\overline{\mu}}_{qt}(\hat{\theta}, \tau, a'-1)d\hat{\theta}\, d\hat{\theta}'\Bigg]. \tag{OA.A.14}$$

The first term captures lines already vacant before the current product-drop decisions, while the second captures active lines whose products are dropped.

Under this approximation, the other lines are independently vacant with probability $q^{\emptyset}_{t+1}(a')$, regardless of the focal state, so

$$p^{e}_{t+1}(a') \equiv \begin{cases} \left[q^{\emptyset}_{t+1}(a')\right]^{n(\varphi)-1}, & n(\varphi) > 1, \\ 1, & n(\varphi) = 1, \end{cases} \qquad a' \geq 2. \tag{OA.A.15}$$

Thus, $p^{e}_{t+1}(a')$ approximates the probability that all non-focal lines are vacant immediately after product-drop decisions. If the focal line is vacant or is itself dropped, $1 - p^{e}_{t+1}(a')$ is the probability that at least one other line remains active and the firm reaches the search stage. No adjustment is required for a focal line in $\mathcal{R}^{cs}_{t+1}$, since its current product remains active through search.

The first adjustment decision of a startup is treated separately. Let $\mathcal{R}^{ns,e}_{t+1}$, $\mathcal{R}^{cs,e}_{t+1}$, and $\mathcal{R}^{ds,e}_{t+1}$ denote the corresponding age-one decision regions for its initial product. For an initially vacant focal line, all non-focal lines are vacant after the drop stage if and only if the startup drops its initial product. Hence,

$$p^{e}_{t+1}(1) \equiv \int_{\mathbb{R}} \mathbf{1}_{\mathcal{R}^{ds,e}_{t+1}}(\hat{\theta}', 1, 1) g(\hat{\theta}' \mid \hat{\theta}^0, 0, 0) d\hat{\theta}'. \tag{OA.A.16}$$

Therefore, $1 - p^{e}_{t+1}(1)$ is the probability that the initial product is retained and the startup reaches the search stage. Unlike (OA.A.15), (OA.A.16) is exact and does not use the conditional-independence approximation.

Continuing product lines. For incumbent product lines transitioning to age $a' > 1$, a line with tenure τ at the end of period t remains active in the same market with tenure $\tau + 1$ if it does not search or if it searches while retaining its product and the search fails:

$$\mu_{p,t+1}(\hat{\theta}', \tau+1, a') = (1-\delta)\left[\mathbf{1}_{\mathcal{R}^{ns}_{t+1}}(\hat{\theta}', \tau+1, a') + (1-x)\mathbf{1}_{\mathcal{R}^{cs}_{t+1}}(\hat{\theta}', \tau+1, a')\right]$$

$$\times \int_{\mathbb{R}} g(\hat{\theta}' \mid \hat{\theta}, \tau, a'-1)\mu_{pt}(\hat{\theta}, \tau, a'-1)d\hat{\theta}. \tag{OA.A.17}$$

For the initial product of a startup transitioning to age one,

$$\begin{aligned}\mu_{p,t+1}(\hat{\theta}', 1, 1) = (1-\delta)\left[\mathbf{1}_{\mathcal{R}^{ns,e}_{t+1}}(\hat{\theta}', 1, 1) + (1-x)\mathbf{1}_{\mathcal{R}^{cs,e}_{t+1}}(\hat{\theta}', 1, 1)\right] \\ \times g(\hat{\theta}' \mid \hat{\theta}^0, 0, 0)\mu_{pt}(\hat{\theta}^0, 0, 0).\end{aligned} \tag{OA.A.18}$$

Newly assigned products. For an incumbent firm transitioning to age $a' > 1$, successful search resets the product state to $(\hat{\theta}^0, 0)$. The resulting inflow into market p is

$$\begin{aligned}\mu_{p,t+1}(\hat{\theta}^0, 0, a') = (1-\delta)\frac{x}{N-1}\sum_{q\neq p}\Bigg\{\sum_{\tau\geq 0}\int_{\mathbb{R}}\int_{\mathbb{R}}\Big[\mathbf{1}_{\mathcal{R}^{cs}_{t+1}}(\hat{\theta}', \tau+1, a') \\ + \left(1-p^e_{t+1}(a')\right)\mathbf{1}_{\mathcal{R}^{ds}_{t+1}}(\hat{\theta}', \tau+1, a')\Big]g(\hat{\theta}' \mid \hat{\theta}, \tau, a'-1)\mu_{qt}(\hat{\theta}, \tau, a'-1)d\hat{\theta}\, d\hat{\theta}' \\ + \left(1-p^e_{t+1}(a')\right)\mu_{qt}(\emptyset, a'-1)\Bigg\}.\end{aligned} \tag{OA.A.19}$$

An active focal line in $\mathcal{R}^{cs}_{t+1}$ reaches search because its existing product is retained. A line in $\mathcal{R}^{ds}_{t+1}$ or an already-vacant line searches only if at least one non-focal line remains active, yielding the factor $1-p^e_{t+1}(a')$.

For startups transitioning to age one, successful search comes either from the initial active line in $\mathcal{R}^{cs,e}_{t+1}$ or from one of the $n(\varphi)-1$ initially vacant lines, which search only when the initial product is retained:

$$\begin{aligned}\mu_{p,t+1}(\hat{\theta}^0, 0, 1) = (1-\delta)\frac{x}{N-1}\sum_{q\neq p}\Bigg[\int_{\mathbb{R}}\mathbf{1}_{\mathcal{R}^{cs,e}_{t+1}}(\hat{\theta}', 1, 1)g(\hat{\theta}' \mid \hat{\theta}^0, 0, 0)d\hat{\theta}' \\ + (n(\varphi)-1)\left(1-p^e_{t+1}(1)\right)\Bigg]\mu_{qt}(\hat{\theta}^0, 0, 0).\end{aligned} \tag{OA.A.20}$$

If the initial product lies in $\mathcal{R}^{ds,e}_{t+1}$, the startup exits before any search takes place.

New operating startups enter at age zero with their initial products at the common prior. Restoring (φ, m) explicitly for this inflow,

$$\mu_{p,t+1}(\hat{\theta}^0, 0, 0, \varphi, m) = \frac{\mathcal{M}^e_{t+1}}{N}\frac{h_\varphi(\varphi)h_m(m \mid \varphi; m_{0,t+1})\mathbf{1}_{\{\varphi\geq\underline{\varphi}^e_{t+1}\}}}{1-H_\varphi(\underline{\varphi}^e_{t+1})}. \tag{OA.A.21}$$

where h_φ and $h_m(\cdot \mid \varphi; m_{0,t+1})$ are the densities associated with H_φ and H_m, respectively, and $\mathcal{M}^e_{t+1}$ denotes the mass of operating entrants. The denominator conditions the entrant type distribution on the operating decision. The corresponding gross mass of startups making the pre-entry investment is $\mathcal{M}^e_{t+1}/[1 - H_\varphi(\underline{\varphi}^e_{t+1})]$.

Destination probabilities. For a line associated with market q, symmetry makes any $p \neq q$ feasible with probability $[N - n(\varphi)]/(N-1)$. By Appendix OA.A.2, its assignment probability conditional on feasibility is $1/[N - n(\varphi)]$. Thus the unconditional probability is

$$\frac{N - n(\varphi)}{N-1}\frac{1}{N - n(\varphi)} = \frac{1}{N-1}.$$

At the startup's first search, excluding only its initial market gives the same probability directly.

Vacant product lines. For incumbent firms transitioning to age $a' > 1$, the mass of vacant lines associated with market p evolves according to

$$\mu_{p,t+1}(\emptyset, a') = (1-\delta)\frac{1-x}{N-1}\sum_{q\neq p}\left(1 - p^e_{t+1}(a')\right)\left[\sum_{\tau\geq 0}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{1}_{\mathcal{R}^{ds}_{t+1}}(\hat{\theta}', \tau+1, a') \times g(\hat{\theta}' \mid \hat{\theta}, \tau, a'-1)\mu_{qt}(\hat{\theta}, \tau, a'-1)d\hat{\theta}\, d\hat{\theta}' + \mu_{qt}(\emptyset, a'-1)\right]. \tag{OA.A.22}$$

The first term inside the brackets captures active lines whose products are dropped and whose subsequent searches fail, while the second captures lines already vacant whose searches fail. Both require the firm to reach the search stage.

For startups transitioning to age one,

$$\mu_{p,t+1}(\emptyset, 1) = (1-\delta)\frac{1-x}{N-1}\sum_{q\neq p}\left(n(\varphi) - 1\right)\left(1 - p^e_{t+1}(1)\right)\mu_{qt}(\hat{\theta}^0, 0, 0). \tag{OA.A.23}$$

This term captures unsuccessful searches by the startup's initially vacant lines conditional on retention of its initial product. If the initial product is dropped, the startup exits before search. There are no market-specific vacant lines at age zero.

In stationary equilibrium, these laws of motion satisfy $\mu_{p,t+1} = \mu_{pt} = \mu_p$ for every product market p.

OA.A.6 Aggregate Search Costs and Price-Index Derivation

Search costs using product-line measures. Applying the portfolio-continuation approximation in Appendix OA.A.5, search costs are as follows; (2.34) gives the exact firm-measure expression. The suppressed states (ψ, φ, m) are integrated over.

$$\begin{aligned}\mathcal{F}_t^s = (1-\delta) f^s \sum_{p=1}^{N} \Bigg[\sum_{a\geq 1} \sum_{\tau \geq 0} \int_{\mathbb{R}} \int_{\mathbb{R}} \Big\{ \mathbf{1}_{\mathcal{R}_t^{cs}}(\hat{\theta}', \tau+1, a+1) + [1 - p_t^e(a+1)]\, \mathbf{1}_{\mathcal{R}_t^{ds}}(\hat{\theta}', \tau+1, a+1) \Big\} \\ \times g(\hat{\theta}' \mid \hat{\theta}, \tau, a) \mu_{p,t-1}(\hat{\theta}, \tau, a) d\hat{\theta}\, d\hat{\theta}' + \sum_{a\geq 1} [1 - p_t^e(a+1)]\, \mu_{p,t-1}(\emptyset, a) \\ + \int_{\mathbb{R}} \Big\{ \mathbf{1}_{\mathcal{R}_t^{cs,e}}(\hat{\theta}', 1, 1) + [n(\varphi) - 1]\, \mathbf{1}_{\mathcal{R}_t^{ns,e} \cup \mathcal{R}_t^{cs,e}}(\hat{\theta}', 1, 1) \Big\} \\ \times g(\hat{\theta}' \mid \hat{\theta}^0, 0, 0) \mu_{p,t-1}(\hat{\theta}^0, 0, 0) d\hat{\theta}' \Bigg]. \end{aligned} \tag{OA.A.24}$$

The first term counts searches by active incumbent lines, the second searches by incumbent vacant lines, and the third searches by age-one startups. In the startup term, the first component corresponds to search by the initial active line, while the second counts searches by its $n(\varphi) - 1$ initially vacant lines. If the initial product lies in $\mathcal{R}_t^{ds,e}$, the startup exits before any search. The factor $1 - \delta$ reflects that search occurs only after the exogenous death shock.

Product-level price index. For a line with end-of-period-$t-1$ state ξ, substitute (2.9) into the CES price index and use $\mathbb{E}[\theta_{ipt}^{(\eta-1)/\eta} \mid \xi] = \mathcal{B}(\xi)$:

$$\begin{aligned} P_{pt}^{1-\eta} &= \int_{\mathcal{X}} \mathbb{E}\left[\theta_{ipt}^{\eta-1} p_{ipt}^{1-\eta} \mid \xi\right] \mu_{p,t-1}(d\xi) \\ &= \left(\frac{\eta}{\eta-1} w_t\right)^{1-\eta} \int_{\mathcal{X}} z(\varphi; z_0)^{\eta-1} \mathcal{B}(\xi)^{\eta-1} \mathbb{E}\left[\theta_{ipt}^{(\eta-1)/\eta} \mid \xi\right] \mu_{p,t-1}(d\xi) \\ &= \left(\frac{\eta}{\eta-1} w_t\right)^{1-\eta} \int_{\mathcal{X}} \mathcal{B}(\xi)^{\eta} z(\varphi; z_0)^{\eta-1} \mu_{p,t-1}(d\xi). \end{aligned}$$

Taking the power $1/(1-\eta)$ yields (2.36). Substituting this index into variable labor demand gives

$$C_{pt} = L_{pt} \left[\int_{\mathcal{X}} \mathcal{B}(\xi)^{\eta} z(\varphi; z_0)^{\eta-1} \mu_{p,t-1}(d\xi) \right]^{1/(\eta-1)}.$$

Symmetry and labor-market clearing then give (2.37).

OA.A.7 Proof of Proposition 3

Proof. Fix true expertise, the observation window, the processing path, and aggregate prices and quantities. Write $\hat{\sigma}^2 \equiv \hat{\sigma}^2_{ip,t-1}$. Expectations average over potential signal histories without conditioning on retention. When permanent expertise is known, the predictive demand term is $\mathcal{B}^{FI}_{ip} = \theta_{ip}^{(\eta-1)/\eta} \exp\{(\eta-1)^2\sigma_\varepsilon^2/(2\eta^2)\}$. Taking the ratio of optimal quantities in (2.7) establishes (2.38); substituting (2.8) yields

$$1 - \omega_{ipt} = \exp\left\{(\eta-1)(\hat{\theta}_{ip,t-1} - \log\theta_{ip}) + \frac{(\eta-1)^2}{2\eta}\hat{\sigma}^2\right\}.$$

By (OA.A.1)–(OA.A.2), $\hat{\theta}_{ip,t-1}$ is conditionally normal. Its exponential moment gives

$$\log\mathbb{E}[1-\omega_{ipt} \mid \theta_{ip}] = \frac{(\eta-1)\hat{\sigma}^2}{\sigma_0^2}\left\{\theta_0 - \log\theta_{ip} + \frac{\eta-1}{2}\left[\left(1+\frac{1}{\eta}\right)\sigma_0^2 - \hat{\sigma}^2\right]\right\}. \tag{OA.A.25}$$

For $\hat{\sigma}^2 > 0$, the expected wedge is positive precisely when the term in braces is negative, establishing (2.39). Lower processing increases $\hat{\sigma}^2$ and lowers the cutoff. Moreover,

$$\frac{\partial\mathbb{E}[\omega_{ipt} \mid \theta_{ip}]}{\partial\hat{\sigma}^2} = \frac{\eta-1}{\sigma_0^2}\mathbb{E}[1-\omega_{ipt} \mid \theta_{ip}]\left\{\log\theta_{ip} - \theta_0 - \frac{\eta-1}{2}\left[\left(1+\frac{1}{\eta}\right)\sigma_0^2 - 2\hat{\sigma}^2\right]\right\}.$$

Under (2.39), the term in braces exceeds $(\eta-1)\hat{\sigma}^2/2 > 0$, so lower processing increases the expected wedge wherever it is positive. □

OA.A.8 Convexity and the Value of Further Learning

We maintain the product-line assumptions of Proposition 1, at firm ages two and above, and hold aggregate prices and quantities fixed. The wage is normalized to one, $d = \beta(1-\delta) \in (0,1)$, and $0 < x < 1$. The prior variance σ_0^2 and observation-noise variance σ_ε^2 are positive and finite, $\eta > 1$, and processing is nonnegative and finite at every finite age. Firm age and the prescribed processing path may continue to change. We write the value functions in demand units, retaining the prior reset and the assigned vacant-line value at endogenous exit used in the main text.

Lemma OA.2 (Convexity and the Value of Further Learning)**.** *At fixed tenure and firm states, the product-line value is continuous, weakly increasing, and convex in the demand term $\mathcal{B}$. The value of retaining without search is strictly increasing and strictly convex. Holding the current demand term and future value function fixed, a larger variance of the next belief update raises expected continuation*

value.

Proof. Put $\alpha = (\eta - 1)/\eta$. For a retained product, normal updating gives

$$\mathcal{B}' = \mathcal{B} \exp\{\alpha \sqrt{v_t^\tau}\, u - \alpha^2 v_t^\tau / 2\}, \qquad v_t^\tau = \hat{\sigma}_{ipt}^2 - \hat{\sigma}_{ip,t+1}^2, \qquad u \sim N(0,1). \tag{OA.A.26}$$

Thus $\mathbb{E}_t \mathcal{B}' = \mathcal{B}$. Expected operating profit is

$$\overline{\pi}_{it+1}(\mathcal{B}, z_i) = \mathcal{A}(z_i) \mathcal{B}^\eta - f^o, \qquad \mathcal{A}(z_i) = \frac{(\eta-1)^{\eta-1}}{\eta^\eta} P^\eta C z_i^{\eta-1} > 0. \tag{OA.A.27}$$

It is strictly increasing and strictly convex in $\mathcal{B} > 0$.

First consider finitely many decision dates and zero continuation thereafter. For each observation, next period's demand term is a positive multiple of the current term, so expectation preserves an increasing, convex continuation value. Adding current expected operating profit makes the no-search value strictly increasing and strictly convex. The continue-and-search value is a positive weighted sum of no-search values, less the search cost, and the drop-and-search value does not depend on the existing product's demand. Taking the maximum preserves monotonicity and convexity. Backward induction gives the asserted properties for every finite horizon.

The payoff bound in the proof of Proposition 1 justifies passage to infinitely many dates. For clarity, the moment used there is $H_{it} = \exp\{(\eta-1)\hat{\theta}_{ipt} + (\eta-1)^2 \hat{\sigma}_{ipt}^2/2\}$. Its conditional expectation is unchanged under retention; a new product starts at the finite prior value H^0. Absolute operating and search payoffs are bounded by a constant times $1 + H_{it}$, and there are at most j additional product starts by date $t + j$. For a finite constant K, independent of the policy and horizon,

$$\begin{aligned} &\sup_{\text{feasible policies}} \mathbb{E}_t[\text{absolute discounted payoffs from date } t+T \text{ onward}] \\ &\quad \le K \sum_{j=T}^{\infty} \beta^j (1 + H_{it} + jH^0) \longrightarrow 0. \end{aligned} \tag{OA.A.28}$$

The bound is uniform on compact sets of the current state. Consequently, the finite-horizon values converge to the infinite-horizon values, preserving continuity, monotonicity, and convexity. Strict convexity and the strictly positive demand derivative of the no-search value follow from current expected profits. This is the recursive approximation argument of Stokey and Lucas Jr (1989, Sections 9.1 and 9.4).

Finally, a larger v_t^τ multiplies the lower-variance demand transition by an independent lognormal

variable with mean one. Conditional Jensen's inequality therefore raises the expectation of every convex continuation value, as in Stokey (2009, Theorem 4.1). The increase is strict when the continuation is strictly convex on a set reached with positive probability. Here sufficiently high demand leads to retaining without search, where strict convexity holds, and the lognormal transition has positive density throughout $(0, \infty)$. □

OA.A.9 Proof of Proposition 4

The information-processing comparison changes accumulated precision while holding future processing fixed. The productivity comparison changes z_i holding the entire processing path fixed, and the tenure comparison changes the observation history at the same firm age. We establish finite information and tenure comparisons directly and report derivative signs where the relevant derivatives exist.

OA.A.9.1 Value matching

Suppressing the firm index on the auxiliary values, let $D_t = V_{it}^{ds}(\emptyset)$. The three choice-specific values satisfy

$$V_{it}^{ns}(\mathcal{B}, \tau) = \beta \overline{\pi}_{it+1}(\mathcal{B}, z_i) + d\mathbb{E}_t V_{i,t+1}(\mathcal{B}', \tau + 1), \tag{OA.A.29}$$

$$V_{it}^{cs}(\mathcal{B}, \tau) = -f^s + (1-x) V_{it}^{ns}(\mathcal{B}, \tau) + x V_{it}^{ns}(\mathcal{B}^0, 0), \tag{OA.A.30}$$

$$D_t = -f^s + x V_{it}^{ns}(\mathcal{B}^0, 0) + (1-x) d D_{t+1}. \tag{OA.A.31}$$

New products begin at the common prior; their tenure resets, but firm age does not. Define

$$U_t(\mathcal{B}, \tau) = V_{it}^{ns}(\mathcal{B}, \tau) - d D_{t+1}, \tag{OA.A.32}$$

$$\Delta_t^V = U_t(\mathcal{B}^0, 0) - f^s / x = (D_t - d D_{t+1})/x. \tag{OA.A.33}$$

Since $V_{it}^{cs} - D_t = (1-x) U_t$ and $V_{it}^{ns} - V_{it}^{cs} = x(U_t - \Delta_t^V)$, the two threshold equations are

$$U_t(\mathcal{B}_{it}^{ds}, \tau) = 0, \qquad U_t(\mathcal{B}_{it}^{cs}, \tau) = \Delta_t^V. \tag{OA.A.34}$$

It is useful to write the value above dropping and searching as

$$F_t(\mathcal{B},\tau) = V_{it}(\mathcal{B},\tau) - D_t = \max\{U_t(\mathcal{B},\tau) - x\Delta_t^V, (1-x)U_t(\mathcal{B},\tau), 0\}. \tag{OA.A.35}$$

Then

$$U_t(\mathcal{B},\tau) = \beta\overline{\pi}_{it+1}(\mathcal{B}, z_i) + d\mathbb{E}_t F_{t+1}(\mathcal{B}', \tau+1). \tag{OA.A.36}$$

The maximum defining F_t is increasing in U_t, decreasing in Δ_t^V, and changes by at most the change in U_t. It is convex in demand by Lemma OA.2. At differentiability points, $U_{t,\mathcal{B}} \geq \beta\mathcal{A}(z_i)\eta\mathcal{B}^{\eta-1} > 0$.

OA.A.9.2 Inherited information and product tenure

Maintain the common product-line assumptions and the definitions of d, D_t, U_t, and Δ_t^V above. Firm age, productivity, aggregate prices, and the search technology are held fixed in the comparisons below. The common future processing path may vary with age. The proof compares finite horizons and shows that the expected contribution of sufficiently distant payoffs vanishes, as in Stokey and Lucas Jr (1989, Sections 9.1 and 9.4), and the convexity implication of the martingale property in Stokey (2009, Theorem 4.1).

Lemma OA.3 (Inherited information with common future processing)**.** *Consider two information histories, H and L, at the same firm age and positive product tenure. Suppose that $\hat{\sigma}_{ipt}^{2,H} < \hat{\sigma}_{ipt}^{2,L}$ and that processing is identical at every date after t. Then $U_t^H(\mathcal{B},\tau) < U_t^L(\mathcal{B},\tau)$ for every $\mathcal{B} > 0$, and both the drop and search thresholds are strictly higher under H.*

Proof. For an existing product retained through date $t+j$, Bayesian updating gives

$$\left(\hat{\sigma}_{ip,t+j}^{2,g}\right)^{-1} = \left(\hat{\sigma}_{ipt}^{2,g}\right)^{-1} + \frac{1}{\sigma_\varepsilon^2}\sum_{k=1}^{j}(1+\psi_{i,t+k}), \qquad g \in \{H, L\}. \tag{OA.A.37}$$

The sum is common across histories. Thus $\hat{\sigma}_{ip,t+j}^{2,H} < \hat{\sigma}_{ip,t+j}^{2,L}$ at every finite j. Moreover, the variance of the next posterior-mean innovation is

$$v_{t+j}^{\tau+j,g} = \hat{\sigma}_{ip,t+j}^{2,g} - \hat{\sigma}_{ip,t+j+1}^{2,g} = \frac{(1+\psi_{i,t+j+1})\left(\hat{\sigma}_{ip,t+j}^{2,g}\right)^2}{\sigma_\varepsilon^2 + (1+\psi_{i,t+j+1})\hat{\sigma}_{ip,t+j}^{2,g}}. \tag{OA.A.38}$$

The last expression is strictly increasing in the current posterior variance. Consequently, $0 < v_{t+j}^{\tau+j,H} < v_{t+j}^{\tau+j,L}$ at every date before replacement.

A new product begins at the common prior, receives its first observation the following period, and thereafter faces the common future processing path. Its value is therefore the same under both histories. The same is true of a vacant line. In particular,

$$D^H_{t+j} = D^L_{t+j}, \qquad \Delta^{V,H}_{t+j} = \Delta^{V,L}_{t+j}, \qquad j \geq 0. \tag{OA.A.39}$$

Let $\alpha = (\eta - 1)/\eta$. The demand transition for a retained product is

$$\mathcal{B}' = \mathcal{B} \exp\left\{ \alpha \sqrt{v^{\tau+j,g}_{t+j}}\, u - \frac{\alpha^2 v^{\tau+j,g}_{t+j}}{2} \right\}, \qquad u \sim N(0,1). \tag{OA.A.40}$$

Hence the transition under L is distributed as the transition under H multiplied by an independent, mean-one lognormal random variable with log variance $\alpha^2(v^{\tau+j,L}_{t+j} - v^{\tau+j,H}_{t+j})$. Conditional Jensen's inequality orders the expectation of every integrable convex function of $\mathcal{B}'$.

To apply this comparison to the infinite-horizon value, first truncate the U recursion after n retained-product transitions, use zero as its terminal value, and keep the common actual values of Δ^V_{t+j} at intermediate dates. At the terminal date the two truncated values coincide. Suppose that the H continuation is no larger than the L continuation. The maximum in the U recursion is increasing in U, and the maximum for the L continuation is convex in demand. Replacing the H continuation by the L continuation weakly raises its expectation; replacing the H innovation variance by the L innovation variance raises it again by conditional Jensen's inequality. Current profits coincide at a fixed demand term. Backward induction therefore orders the truncated values at every date.

This ordering passes to the infinite horizon. The maximum in the U recursion is 1-Lipschitz in U. Consequently, the difference between the actual value and its truncated recursion is at most d^n times the expected absolute terminal value along the hypothetical path on which the existing product is always retained. The payoff bound used for Proposition 1 gives a constant $C < \infty$, independent of the truncation date, such that this error is at most $Cd^n\mathbb{E}_t[1 + \mathcal{B}^\eta_{t+n}]$. Along that path,

$$\mathbb{E}_t \mathcal{B}^\eta_{t+n} = \mathcal{B}^\eta \exp\left\{ \frac{\eta(\eta-1)\alpha^2}{2} \left(\hat{\sigma}^{2,g}_{ipt} - \hat{\sigma}^{2,g}_{ip,t+n} \right) \right\} \leq \mathcal{B}^\eta \exp\left\{ \frac{\eta(\eta-1)\alpha^2\sigma_0^2}{2} \right\}. \tag{OA.A.41}$$

Since $d < 1$, the truncation error vanishes. Thus $U^H_t(\mathcal{B}, \tau) \leq U^L_t(\mathcal{B}, \tau)$.

The inequality is strict. Use the continuation value F^L_{t+1} defined in (OA.A.35), with next-period tenure understood. For sufficiently large $\mathcal{B}'$, no search is optimal and $F^L_{t+1} = U^L_{t+1} - x\Delta^V_{t+1}$. This

function is strictly convex there because the no-search value is strictly convex. The nondegenerate lognormal transition reaches that region with positive probability. Applying strict conditional Jensen's inequality to F^L_{t+1}, while using the weak ordering of the two endogenous continuations, yields $U^H_t(\mathcal{B},\tau) < U^L_t(\mathcal{B},\tau)$ for every $\mathcal{B} > 0$. Both values are strictly increasing in demand, and the two threshold equations have the same right-hand sides, 0 and Δ^V_t. Their unique solutions therefore satisfy $\mathcal{B}^{ds,H}_{it} > \mathcal{B}^{ds,L}_{it}$ and $\mathcal{B}^{cs,H}_{it} > \mathcal{B}^{cs,L}_{it}$. □

Information-processing part of Proposition 4. Increase ψ_{it} holding processing at every other date fixed. For an existing product of positive tenure, current processing enters its inherited precision with coefficient $1/\sigma^2_\varepsilon$. Until replacement,

$$\begin{aligned} \frac{\partial \hat{\sigma}^2_{ip,t+j}}{\partial \psi_{it}} &= -\frac{\left(\hat{\sigma}^2_{ip,t+j}\right)^2}{\sigma^2_\varepsilon} < 0, \\ \frac{\partial v^{\tau+j}_{t+j}}{\partial \psi_{it}} &= -\frac{\left(\hat{\sigma}^2_{ip,t+j}\right)^2 - \left(\hat{\sigma}^2_{ip,t+j+1}\right)^2}{\sigma^2_\varepsilon} < 0, \qquad j \geq 0. \end{aligned} \tag{OA.A.42}$$

Replacement opportunities are unchanged: $\partial_{\psi_{it}} \Delta^V_{t+j} = 0$ for all $j \geq 0$. Lemma OA.3 gives strict finite increases in both thresholds.

For the strict derivative signs, consider a positive increment in ψ_{it} and freeze the next-period continuation at its value before the increment. The endogenous continuation after the increment is weakly lower by Lemma OA.3. The increase in the current value is therefore bounded above by the effect of the smaller current innovation variance on the frozen continuation. This latter effect has a strictly negative derivative. To see this directly, let F denote that convex continuation and $p_v(y \mid \mathcal{B})$ the density in (OA.A.40). Gaussian differentiation and integration by parts give

$$\frac{\partial}{\partial v}\mathbb{E}_v F(\mathcal{B}') = \frac{\alpha^2}{2}\int_0^\infty y^2 p_v(y \mid \mathcal{B})\, F''(dy) > 0. \tag{OA.A.43}$$

Here F'' is the nonnegative second-derivative measure of the convex function F; the formula also covers its kinks. The bound on expected payoffs and the finite moments of the lognormal demand term justify differentiation. Strict positivity follows from strict convexity in the no-search region and the positive transition density there. Combining (OA.A.43) with (OA.A.42) gives a strictly negative upper right difference quotient for U_t. Thus, wherever the parameter derivative exists, $U_{t,\psi_{it}}(\mathcal{B},\tau) < 0$.

At points where the threshold functions are differentiable, their defining equations and $U_{t,\mathcal{B}} > 0$

imply

$$\frac{\partial \mathcal{B}_{it}^{j}}{\partial \psi_{it}} = -\frac{U_{t,\psi_{it}}(\mathcal{B}_{it}^{j}, \tau)}{U_{t,\mathcal{B}}(\mathcal{B}_{it}^{j}, \tau)} > 0, \qquad j \in \{ds, cs\}. \tag{OA.A.44}$$

The argument conditions on the entire future processing path; it requires neither a stationary future nor a restriction on the effective horizon beyond $d < 1$. □

Product-tenure part of Proposition 4. Fix firm age and the complete processing path, and compare two feasible product tenures, τ and $\tau + 1$. The longer-tenure product has the additional observation dated $t - \tau$. Accordingly,

$$\frac{1}{\hat{\sigma}_{ipt}^{2}(\tau + 1)} - \frac{1}{\hat{\sigma}_{ipt}^{2}(\tau)} = \frac{1 + \psi_{i,t-\tau}}{\sigma_{\varepsilon}^{2}} > 0. \tag{OA.A.45}$$

The two products face identical future processing and replacement opportunities. The proof of Lemma OA.3 depends on tenure only through inherited precision, and therefore applies to these two states. It gives $U_t(\mathcal{B}, \tau + 1) < U_t(\mathcal{B}, \tau)$ at every fixed $\mathcal{B} > 0$. The unique solutions to the two threshold equations satisfy

$$\mathcal{B}_{it}^{ds}(\tau + 1) > \mathcal{B}_{it}^{ds}(\tau), \qquad \mathcal{B}_{it}^{cs}(\tau + 1) > \mathcal{B}_{it}^{cs}(\tau). \tag{OA.A.46}$$

□

These comparisons also raise the thresholds expressed in posterior-mean units. Inverting the demand term gives

$$\hat{\theta}_{ipt}^{j} = \frac{\log \mathcal{B}_{it}^{j}}{\alpha} - \frac{\alpha}{2}\left(\hat{\sigma}_{ipt}^{2} + \sigma_{\varepsilon}^{2}\right), \qquad j \in \{ds, cs\}. \tag{OA.A.47}$$

In each comparison the demand threshold rises and posterior variance falls, so both terms increase the posterior-mean threshold.

OA.A.9.3 Productivity and expected fixed costs

Hold the information-processing history and future path, product scope, search technology, fixed costs, and aggregate prices and quantities fixed. Normalize the wage to one and put $d = \beta(1 - \delta)$. The comparisons below apply to the ordinary product-line decisions and use the maintained vacant-line continuation value at endogenous firm exit.

Policy costs and the productivity envelope. Let μ specify product-switching decisions and the decisions used to evaluate the assigned vacant-line continuation at endogenous exit. For initial

decision $j \in \{ns, ds\}$, let $I^{o,\mu}_{t+\ell+1}$ indicate production determined at date $t+\ell$, and let $I^{s,\mu}_{t+\ell}$ indicate search at that date. Its expected discounted fixed-cost burden is

$$\mathcal{K}^{j,\mu}_{it}(\mathcal{B},\tau) = \mathbb{E}^{j,\mu}_t \sum_{\ell=0}^{\infty} d^\ell \left[\beta f^o I^{o,\mu}_{t+\ell+1} + f^s I^{s,\mu}_{t+\ell}\right]. \tag{OA.A.48}$$

The sum follows the product line through subsequent product replacements. The indicators are evaluated conditional on no exogenous exit before the corresponding decision date; the factor d^ℓ already includes that survival probability. At endogenous firm exit, the cost burden includes the fixed-cost component of the assigned vacant-line continuation. Equivalently, if $\mathcal{T}$ is the number of decision dates until that exit, the sum in (OA.A.48) can be stopped before $\mathcal{T}$ and augmented by

$$\mathbf{1}\{\mathcal{T} < \infty\} d^{\mathcal{T}} \mathcal{K}^{ds,\mu}_{t+\mathcal{T}}.$$

Thus the accounting uses the same continuation convention as V^{ns} and V^{ds}.

The factor β on f^o reflects next-period production, whereas search is paid for at the current decision date. Initial no search therefore incurs βf^o, while initial drop and search incurs $f^s + x\beta f^o$ in expectation. Subsequent decision dates are discounted by d. In particular, the expected discounted fixed costs from date $t+T$ onward are bounded under every policy by

$$\frac{(\beta f^o + f^s) d^T}{1-d}. \tag{OA.A.49}$$

After optimizing production quantities, a fixed policy has value

$$V^{j,\mu}_{it} = z_i^{\eta-1} \mathcal{G}^{j,\mu}_{0,it} - \mathcal{K}^{j,\mu}_{it}, \tag{OA.A.50}$$

where $\mathcal{G}^{j,\mu}_{0,it}$ is the expected discounted sum of gross operating payoffs, divided by $z_i^{\eta-1}$. Gross operating payoffs are revenues less variable production costs, including the same component of the assigned vacant-line continuation. The policy space and both coefficients are independent of z_i under the stated comparison. Let $\mathcal{K}^j_{it}$ denote (OA.A.48) under an optimal continuation for initial decision j. At regular parameter values, the envelope theorem gives

$$\frac{z_i}{\eta-1} V^j_{it,z_i} = V^j_{it} + \mathcal{K}^j_{it}. \tag{OA.A.51}$$

The derivative identity applies where the values are differentiable. On a compact productivity in-

terval, the derivative of a fixed policy's gross operating payoffs is bounded by a constant times those payoffs. Thus (OA.A.28) also makes the expected contribution of distant payoff derivatives vanish. Future demand is continuously distributed, so switching indifferences occur with probability zero; changing optimal decisions adds no term to the envelope identity. Together with (OA.A.49), these bounds justify using the identity in the infinite-horizon problem.

Exact threshold derivatives. Use

$$U_t(\mathcal{B}, \tau) = V_{it}^{ns}(\mathcal{B}, \tau) - dD_{t+1}, \qquad \Delta_t^V = U_t(\mathcal{B}^0, 0) - f^s/x,$$

where $D_t = V_{it}^{ds}(\emptyset)$. Applying (OA.A.51) and differentiating $U_t(\mathcal{B}_{it}^{ds}, \tau) = 0$ and $U_t(\mathcal{B}_{it}^{cs}, \tau) = \Delta_t^V$ gives

$$\partial_{z_i} \mathcal{B}_{it}^{ds} = \frac{\eta - 1}{z_i} \frac{d\mathcal{K}_{t+1}^{ds} - \mathcal{K}_t^{ns}(\mathcal{B}_{it}^{ds}, \tau)}{U_{t,\mathcal{B}}(\mathcal{B}_{it}^{ds}, \tau)}, \tag{OA.A.52}$$

$$\partial_{z_i} \mathcal{B}_{it}^{cs} = \frac{\eta - 1}{z_i} \frac{f^s/x + \mathcal{K}_t^{ns}(\mathcal{B}^0, 0) - \mathcal{K}_t^{ns}(\mathcal{B}_{it}^{cs}, \tau)}{U_{t,\mathcal{B}}(\mathcal{B}_{it}^{cs}, \tau)}. \tag{OA.A.53}$$

Consequently, the respective strict cost orderings are

$$\mathcal{K}_t^{ns}(\mathcal{B}_{it}^{ds}, \tau) > d\mathcal{K}_{t+1}^{ds}, \tag{OA.A.54}$$

$$\mathcal{K}_t^{ns}(\mathcal{B}_{it}^{cs}, \tau) - \mathcal{K}_t^{ns}(\mathcal{B}^0, 0) < f^s/x. \tag{OA.A.55}$$

They imply $\partial_{z_i} \mathcal{B}_{it}^{ds} < 0$ and $\partial_{z_i} \mathcal{B}_{it}^{cs} > 0$. When both hold, the distance between the thresholds increases with productivity.

These inequalities have a direct interpretation in terms of gross operating payoffs at the two indifference points. At the drop threshold, the common search and successful-replacement terms cancel, leaving $V_{it}^{ns} = dD_{t+1}$. A larger fixed-cost burden for retaining the product following an unsuccessful search therefore means greater gross operating payoffs. Higher productivity scales these payoffs and favors retention, lowering the demand required to retain a product. At the search threshold,

$$V_{it}^{ns}(\mathcal{B}^0, 0) - V_{it}^{ns}(\mathcal{B}_{it}^{cs}, \tau) = f^s/x.$$

Condition (OA.A.55) says that the new product has greater gross operating payoffs at this indifference point. Higher productivity increases the gain from a successful replacement and raises the

search threshold.

The cost orderings concern optimal continuations. They are exact local sign conditions at regular parameter values, not restrictions directly on primitives. Strict finite comparisons follow on productivity intervals where the corresponding orderings hold at the thresholds throughout the interval.

Uniform primitive sufficient restrictions. The following result gives one simple set of primitive restrictions for any deterministic information-processing history and future path. Write

$$L = 1 - d(1 - x), \qquad r = \frac{f^s}{\beta f^o}.$$

Proposition OA.1 (Primitive productivity restrictions)**.** *Suppose* $0 < d < 1$*,* $0 < x < 1$*, positive fixed costs, and the maintained product-line conditions. If*

$$d < \frac{1}{1 + x}, \tag{OA.A.56}$$

then the separate sufficient restrictions are

$$r < \frac{1 - d}{dL} \quad \Longrightarrow \quad \partial_{z_i} \mathcal{B}_{it}^{ds} < 0, \tag{OA.A.57}$$

$$r > \frac{dx(1 - x)}{L} \quad \Longrightarrow \quad \partial_{z_i} \mathcal{B}_{it}^{cs} > 0. \tag{OA.A.58}$$

Both signs hold strictly between these bounds, which define a nonempty interval under (OA.A.56)*.*

Proof. Put $a_0 = 1 - d$ and $C_o = \beta f^o$. For this proof define

$$J_t(\mathcal{B}, \tau) = \mathcal{K}_t^{ns}(\mathcal{B}, \tau) - d\mathcal{K}_{t+1}^{ds}, \qquad E_t = J_t(\mathcal{B}^0, 0) + f^s/x.$$

The signs in (OA.A.54)–(OA.A.55) require $J_t > 0$ and $E_t - J_t > 0$, respectively, at the relevant thresholds. Differentiating the relative-value recursion, or collecting its fixed-cost terms under each optimal decision, yields

$$J_t = C_o + d\mathbb{E}_t \left[p_{t+1} J_{t+1} - x\mathbf{1}\{ns \text{ at } t+1\} E_{t+1} \right], \tag{OA.A.59}$$

where p_{t+1} equals 1, $1 - x$, or 0 for no search, continue and search, or drop and search. Normalize $j_t = J_t/C_o$ and $e_t = E_t/C_o = j_t(\mathcal{B}^0, 0) + r/x$. The three possible next-date contributions in (OA.A.59)

are

$$j - xj_0 - r, \qquad (1-x)j, \qquad 0,$$

where j_0 refers to a new product at that date. These identities remain valid in a truncated problem regardless of the sign of its terminal replacement value.

Condition (OA.A.56) is equivalent to $a_0 > dx$. Let $r_0 = dx(1-x)/L$. For $r \geq r_0$, set

$$\overline{j} = 1/L, \qquad \underline{j} = \min\{1, \overline{j} - dr/a_0\}.$$

Then $\underline{j} \leq 1 \leq \overline{j}$ and $\underline{j} + r/x \geq \overline{j}$, strictly if $r > r_0$. If $\underline{j} = 1$, the latter difference is $r/x - d(1-x)/L$; otherwise it is $r(1/x - d/a_0) > 0$.

Suppose all next-date normalized costs lie in $[\underline{j}, \overline{j}]$. Their contributions lie in

$$[\underline{j} - x\overline{j} - r, \overline{j} - x\underline{j} - r], \qquad [(1-x)\underline{j}, (1-x)\overline{j}], \qquad \{0\}.$$

Adding one after multiplying by d preserves the upper endpoint because $1 + d(1-x)\overline{j} = \overline{j}$ and $\overline{j} - x\underline{j} - r \leq (1-x)\overline{j}$. It preserves the lower endpoint because

$$1 + d(\underline{j} - x\overline{j} - r) - \underline{j} = a_0(\overline{j} - dr/a_0 - \underline{j}) \geq 0,$$
$$1 + d(1-x)\underline{j} - \underline{j} = 1 - L\underline{j} \geq 0,$$

and $1 \geq \underline{j}$. The terminal no-search cost is $j = 1$, so backward induction gives this invariant interval. It is valid even if $\underline{j} < 0$. For $r > r_0$, it yields

$$e_t - j_t \geq \underline{j} + r/x - \overline{j} > 0,$$

which establishes the search sign without imposing an upper bound on r. If also $r < a_0/(dL)$, then $\underline{j} > 0$, giving the drop sign.

For $r \leq r_0$, use instead

$$\underline{j} = 1, \qquad \overline{j}' = (1 - dx - dr)/a_0 \geq 1/L.$$

The upper no-search recursion is an equality. The lower one preserves one because

$$1 - x\bar{j}' - r = (a_0 - dx)(1 - x - r)/a_0 \geq 0,$$

where $r \leq r_0 < 1 - x$. The other two decisions preserve the interval directly. Hence $J_t \geq C_o > 0$ for this cost range too, proving (OA.A.57) without a lower restriction on r.

The finite-horizon bounds pass to infinite-horizon cost envelopes using (OA.A.49) and the bound on expected payoffs. Alternatively, the original values are suprema of affine functions of $z_i^{\eta-1}$; local uniform convergence and almost-everywhere envelope differentiation pass the bounds to the limiting values, and integration gives finite comparisons. Finally, $a_0 > dx > d^2x(1-x)$ implies

$$\frac{dx(1-x)}{L} < \frac{a_0}{dL},$$

so the joint interval is nonempty. □

These are uniform sufficient bounds on future fixed-cost burdens. Their failure does not determine either threshold's direction. For example, the reference values $d \simeq 0.928$ and $x = 0.335$ fail (OA.A.56), which requires $d < 0.7491$. Results obtained by making processing constant over time do not replace this restriction for the general age-dependent model.

OA.A.10 Accumulated Information and Future Learning

Information processing affects product switching through the precision of accumulated information and the informativeness of future signals. The comparison in Proposition 4(i) isolates the first channel. We now hold the current posterior fixed and vary future processing, then combine the two changes. Changes in replacement opportunities are part of the future learning channel. Throughout, maintain the product-line assumptions above and hold productivity, aggregate conditions, the prior, search technology, and fixed costs fixed. Derivative statements apply wherever the relevant derivatives exist, with right derivatives at zero future processing.

OA.A.10.1 The two information comparisons

For the complementary future-learning comparison, let processing be constant at $\bar{\psi}$ from date $t+1$ onward. Write

$$k = \frac{\sigma_\varepsilon^2}{\hat{\sigma}_{ipt}^2} = c + \tau + \sum_{s=t-\tau+1}^{t} \psi_{is}, \qquad q = 1 + \bar{\psi}, \qquad c = \frac{\sigma_\varepsilon^2}{\sigma_0^2}. \tag{OA.A.60}$$

Here k is accumulated precision in units of signal precision; a new product starts at c. After j additional observations, the existing product's posterior variance is $\sigma_\varepsilon^2/(k+jq)$, and its next posterior-mean innovation has variance

$$v_j(k,q) = \frac{\sigma_\varepsilon^2 q}{(k+jq)[k+(j+1)q]}, \qquad j \geq 0. \tag{OA.A.61}$$

As above, $\mathcal{B}' = \mathcal{B}\exp\{\alpha\sqrt{v_0(k,q)}Z - \alpha^2 v_0(k,q)/2\}$, with $\alpha = (\eta-1)/\eta$ and $Z \sim N(0,1)$.

The accumulated information comparison raises k at a fixed current demand term and common future processing path. Lemma OA.3 and the derivative argument above establish $U_k < 0$ and strictly higher thresholds: every subsequent innovation contracts, while replacement opportunities are unchanged. The future learning comparison instead raises q while holding the current posterior mean and variance fixed, hence holding both k and $\mathcal{B}$ fixed. Each future processing level may be required to be at least current processing to preserve a nondecreasing age profile. In this comparison,

$$\begin{aligned} \partial_q v_j &= \frac{\sigma_\varepsilon^2[k^2 - j(j+1)q^2]}{(k+jq)^2[k+(j+1)q]^2}, \\ \sum_{j=0}^{n-1} v_j &= \sigma_\varepsilon^2\left(\frac{1}{k} - \frac{1}{k+nq}\right). \end{aligned} \tag{OA.A.62}$$

The next innovation becomes more variable, whereas sufficiently distant innovations become less variable as information accumulates faster. Cumulative learning through every fixed future horizon increases. Thus, the accumulated-information argument cannot simply be reversed for future learning.

OA.A.10.2 Learning about the existing product and replacement opportunities

Since new products receive their first observations next period, their values depend only on the future processing level. The vacant-line value is therefore stationary, and $\Delta^V(q) = (1-d)D(q)/x > 0$. To

separate the two effects of future learning, treat $\Delta > 0$ as an independent argument of the relative retention value:

$$U(\mathcal{B}, k; q, \Delta) = \beta\overline{\pi}(\mathcal{B}, z) + d\mathbb{E}F(\mathcal{B}', k+q; q, \Delta), \tag{OA.A.63}$$

$$F(\mathcal{B}, k; q, \Delta) = \max\{U(\mathcal{B}, k; q, \Delta) - x\Delta,\ (1-x)U(\mathcal{B}, k; q, \Delta),\ 0\}. \tag{OA.A.64}$$

Retaining without search incurs the deduction $x\Delta$; retaining while searching ends the existing product with probability x; dropping ends it immediately. This auxiliary problem isolates learning about the existing product. It coincides with the actual relative-value recursion when $\Delta = \Delta^V(q)$, which satisfies

$$\Delta^V(q) + f^s/x = U(\mathcal{B}^0, c; q, \Delta^V(q)). \tag{OA.A.65}$$

The actual thresholds solve

$$\begin{aligned} U(\mathcal{B}^{ds}, k; q, \Delta^V(q)) &= 0, \\ U(\mathcal{B}^{cs}, k; q, \Delta^V(q)) &= \Delta^V(q), \end{aligned} \tag{OA.A.66}$$

with $U_{\mathcal{B}} \geq \beta\overline{\pi}_{\mathcal{B}} > 0$ at both thresholds.

Lemma OA.4 (The two effects of future learning)**.** *At fixed Δ, higher future processing strictly raises U; at fixed q, better replacement opportunities strictly lower U. At regular parameter values, define*

$$\mathcal{L} = \partial_q U|_{k,\Delta} > 0, \qquad \mathcal{W} = -U_\Delta > 0. \tag{OA.A.67}$$

These sensitivities depend on the current product state. If I_j^{ns} indicates that the original product remains and no search is chosen at date $t+j$, then

$$\mathcal{W} = x\mathbb{E}\sum_{j=1}^{\infty} d^j I_j^{ns}, \qquad 0 < \mathcal{W} \leq \frac{xd}{1-d}. \tag{OA.A.68}$$

The indicator is conditional on survival from exogenous exit, whose probability is included in d^j. In the actual stationary-future problem, replacement opportunities strictly improve:

$$\Delta_q^V = \frac{\mathcal{L}(\mathcal{B}^0, c)}{1 + \mathcal{W}(\mathcal{B}^0, c)} > 0, \tag{OA.A.69}$$

where both sensitivities are evaluated at $\Delta^V(q)$.

Proof. For $q_H > q_L$, add independent noise to the high-precision observations to reproduce the

low-precision signal sequence. Use the lower-information product decisions and search outcomes, while choosing quantities with the additional information. The deductions $x\Delta$, dropping dates, and replacement probabilities are unchanged, and conditional expected gross operating payoffs weakly increase. Hence the auxiliary value weakly increases.

The production gain after the first future observation makes the comparison strict. Let $s'_g = \sigma_\varepsilon^2/(k+q_g)$ and $\chi = \eta(\eta-1)\alpha^2/2 > 0$. Conditional on the lower-information posterior,

$$\mathbb{E}[(\mathcal{B}'_H)^\eta \mid \mathcal{I}'_L] = (\mathcal{B}'_L)^\eta \exp\{\chi(s'_L - s'_H)\}. \tag{OA.A.70}$$

On the positive-probability event $\overline{\pi}(\mathcal{B}'_L, z) > 0$, the lower-information policy retains the existing product with probability at least $1-x$. Put $M = \mathbb{E}[(\mathcal{B}'_L)^\eta \mathbf{1}\{\overline{\pi}(\mathcal{B}'_L, z) > 0\}] > 0$. The feasible-policy gain is at least $d\beta(1-x)\mathcal{A}(z)M\{e^{\chi(s'_L - s'_H)} - 1\}$, so its local limit gives

$$\mathcal{L} \geq d\beta(1-x)\mathcal{A}(z)M\chi\frac{\sigma_\varepsilon^2}{(k+q)^2} > 0. \tag{OA.A.71}$$

The payoff bound in (OA.A.28) justifies the same comparison at the infinite horizon.

An increase in Δ raises the deduction on each future no-search decision while the original product remains. The envelope theorem yields (OA.A.68); the sum starts at one because today's action in U is fixed and carries no such deduction. The bound follows from $I_j^{ns} \leq 1$. A sufficiently high next demand term leads to no search with positive probability, making $\mathcal{W} > 0$. Future switching ties have probability zero, and the discounted bound justifies differentiation. For a finite increase $\Delta_H > \Delta_L$, using a policy optimal at Δ_H under Δ_L raises its payoff by $(\Delta_H - \Delta_L)x\mathbb{E}_H \sum_{j\geq 1} d^j I_j^{ns} > 0$. Thus U strictly decreases without requiring differentiability everywhere.

Differentiating (OA.A.65) gives $(1+\mathcal{W}(\mathcal{B}^0, c))\Delta_q^V = \mathcal{L}(\mathcal{B}^0, c)$. For finite comparisons, the left side of the fixed-point equation increases in Δ, while its right side decreases in Δ and increases in q. Its solution therefore strictly increases with q. □

Proposition OA.2 (Future learning and product-switching thresholds)**.** *Holding the current posterior fixed, higher stationary future processing changes the thresholds according to*

$$\partial_q \mathcal{B}^{ds} = \frac{\mathcal{W}(\mathcal{B}^{ds})\Delta_q^V - \mathcal{L}(\mathcal{B}^{ds})}{U_{\mathcal{B}}(\mathcal{B}^{ds})}, \tag{OA.A.72}$$

$$\partial_q \mathcal{B}^{cs} = \frac{[1+\mathcal{W}(\mathcal{B}^{cs})]\Delta_q^V - \mathcal{L}(\mathcal{B}^{cs})}{U_{\mathcal{B}}(\mathcal{B}^{cs})}. \tag{OA.A.73}$$

Each sign is the sign of its numerator. These are exact local conditions on value sensitivities, rather than primitive restrictions.

Proof. At fixed k, the actual relative-value derivative is $dU/dq = \mathcal{L} - \mathcal{W}\Delta_q^V$. Differentiating (OA.A.66) gives the two expressions because $U_{\mathcal{B}} > 0$. □

Better future learning about the existing product, holding replacement opportunities fixed, lowers both thresholds by increasing the value of further experimentation. Better replacement opportunities, holding that learning technology fixed, raise both thresholds. The search comparison includes the direct gain Δ_q^V as well as the effect on future retention decisions. Because the sensitivities are evaluated at different thresholds, this additional term alone orders neither their derivatives nor the change in their distance. The actual future learning channel combines these opposing effects.

OA.A.10.3 A short effective horizon

A sufficiently short effective horizon yields an analytical net-sign result. Hold β, all other primitives except δ, and the current posterior fixed; let $\delta \uparrow 1$, so the effective continuation factor $d = \beta(1-\delta)$ approaches zero. Suppose $\Delta_0 = \beta\overline{\pi}(\mathcal{B}^0, z) - f^s/x > 0$. At $d = 0$, the thresholds and continuation value are

$$b^{ds} = (f^o/\mathcal{A})^{1/\eta}, \qquad b^{cs} = [(\mathcal{B}^0)^\eta - f^s/(x\beta\mathcal{A})]^{1/\eta}, \tag{OA.A.74}$$
$$F_0(\mathcal{B}) = (1-x)[\beta\overline{\pi}(\mathcal{B}, z)]_+ + x[\beta\overline{\pi}(\mathcal{B}, z) - \Delta_0]_+.$$

For inherited variance $s > 0$, define

$$v(s,q) = s - \frac{s\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + sq}, \qquad H(\mathcal{B}, v) = \mathbb{E}F_0(\mathcal{B}e^{\alpha\sqrt{v}Z - \alpha^2 v/2}),$$
$$J(\mathcal{B}, s; q) = H_v(\mathcal{B}, v(s,q))\frac{s^2\sigma_\varepsilon^2}{(\sigma_\varepsilon^2 + sq)^2} > 0. \tag{OA.A.75}$$

This expectation and its variance derivative are given by the normal-CDF formulas in (OA.A.99); no dynamic solution is required.

Corollary OA.1 (Future learning with a short effective horizon)**.** *Under $\Delta_0 > 0$, at a fixed positive inherited variance $s = \hat{\sigma}^2_{ipt}$ and fixed $q \geq 1$,*

$$\partial_q \mathcal{B}^{ds} = -d \frac{J(b^{ds}, s; q)}{\beta \overline{\pi}_{\mathcal{B}}(b^{ds}, z)} + o(d), \tag{OA.A.76}$$

$$\partial_q \mathcal{B}^{cs} = d \frac{J(\mathcal{B}^0, \sigma_0^2; q) - J(b^{cs}, s; q)}{\beta \overline{\pi}_{\mathcal{B}}(b^{cs}, z)} + o(d). \tag{OA.A.77}$$

Future processing therefore lowers the drop threshold for all sufficiently small $d > 0$. A strictly positive or negative difference in the search numerator determines its sign for all sufficiently small $d > 0$.

Proof. As $d \downarrow 0$, $U \to \beta\overline{\pi}$, $\Delta^V \to \Delta_0$, and the roots converge to (OA.A.74). The leading effect of future processing is the first future observation, giving $\mathcal{L}(\mathcal{B}, k) = dJ(\mathcal{B}, s; q) + o(d)$, locally in the current state and q. To justify this derivative limit, first use finite horizons and integrate the continuation maximum against the smooth Gaussian transition. The limiting thresholds have positive demand slopes, and Gaussian states hit their boundaries with probability zero. The local moment bounds underlying (OA.A.28) permit dominated differentiation; contributions from dates two and later are bounded by a constant times $\sum_{j \geq 2}(j+1)d^j = O(d^2)$. Thus the first-step derivative converges to J as the horizon grows and d falls. Equation (OA.A.68) gives $\mathcal{W} = O(d)$, and (OA.A.69) gives $\Delta^V_q = dJ(\mathcal{B}^0, \sigma_0^2; q) + o(d)$. Substitute these expressions into (OA.A.72)–(OA.A.73) and use $U_{\mathcal{B}} \to \beta\overline{\pi}_{\mathcal{B}}$ at the limiting roots. □

The existing product's learning gain affects the drop margin at first order in d, while the effect of improved replacement opportunities on its future decisions is of higher order. Both learning gains enter the search margin at first order. The corollary supplies neither an explicit upper bound on d nor a conclusion for the reference calibration.

OA.A.10.4 Combined changes with a stationary future

An improvement in the processing path generally changes both accumulated precision and future learning. Let $k(\lambda)$ and $q(\lambda)$ describe such a comparison, with $k', q' \geq 0$ and $k' + q' > 0$. The parameter λ indexes the comparison, rather than a new firm characteristic.

Corollary OA.2 (Combined improvement with a stationary future)**.** *At regular parameter values, the threshold changes decompose as*

$$\frac{d\mathcal{B}^j}{d\lambda} = \partial_k \mathcal{B}^j \, k' + \partial_q \mathcal{B}^j \, q', \qquad j \in \{ds, cs\}. \tag{OA.A.78}$$

Both derivatives are strictly positive under the primitive restriction

$$k^2 q' \leq q k' (2k + q). \tag{OA.A.79}$$

For a finite comparison, it suffices that this restriction hold along the comparison interval.

Proof. The chain rule gives (OA.A.78). Along the retained-product history,

$$\frac{dv_j}{d\lambda} = \frac{\sigma_\varepsilon^2 \{q'[k^2 - j(j+1)q^2] - qk'[2k + (2j+1)q]\}}{(k+jq)^2[k+(j+1)q]^2}. \tag{OA.A.80}$$

The numerator decreases with j. Condition (OA.A.79) makes it nonpositive at $j = 0$ and strictly negative thereafter. Replacement opportunities weakly improve by Lemma OA.4, strictly if $q' > 0$. The local comparison in Proposition OA.3, proved below, therefore gives strict threshold derivative signs, including equality in (OA.A.79). Its regularity condition holds locally for these constant-future changes. The finite comparison follows from the same innovation and opportunity ordering along the interval, or directly from the finite version of that proposition. □

The restriction requires accumulated precision to increase sufficiently relative to future signal precision that every remaining belief revision contracts. For constant processing throughout the history, $k = c + \tau q$ and $q = 1 + \psi$, so it becomes $(1+\psi)\sqrt{\tau(\tau+1)} \geq c$. For a proportional history followed by a stationary future, $k = C_* + \omega q$ gives $q\sqrt{\omega(\omega+1)} \geq C_*$. These are the stationary specializations below. For a general age-dependent future path, replacement opportunities need not be ordered; the next subsection states the additional conditions required for the same argument.

OA.A.11 Changes in the Processing History and Future Processing

Maintain the product-line assumptions and thresholds of Proposition 1, and consider ordinary firm ages two and above. Throughout this subsection, productivity, the prior, scope, prices, search technology, and fixed costs are held fixed. A comparison holds the current demand term $\mathcal{B}$ fixed when comparing continuation values. Let $d = \beta(1-\delta)$, retain U_t and Δ_t^V from (OA.A.33), and write the

continuation maximum as

$$F_t(\mathcal{B},\tau) = \max\{U_t(\mathcal{B},\tau) - x\Delta_t^V, (1-x)U_t(\mathcal{B},\tau), 0\}. \tag{OA.A.81}$$

The learning-value lemma gives convexity in $\mathcal{B}$, with strict curvature on a high-demand interval. For two processing histories and future paths, indexed by H and L, let $v_{t+j}^{g,\tau+j}$ be the next innovation variance of the existing product if it has been retained, where $g \in \{H, L\}$.

Proposition OA.3 (General processing-path comparison)**.** *Suppose*

$$v_{t+j}^{H,\tau+j} \le v_{t+j}^{L,\tau+j} \qquad \textit{for every } j \ge 0, \tag{OA.A.82}$$

$$\Delta_{t+j}^{V,H} \ge \Delta_{t+j}^{V,L} \qquad \textit{for every } j \ge 1. \tag{OA.A.83}$$

Then the product-drop threshold is weakly higher under H. If also $\Delta_t^{V,H} \ge \Delta_t^{V,L}$, the search threshold is weakly higher under H. Both applicable comparisons are strict if the variance inequality is strict at at least one finite future date, including the current date.

For the local version, assume the family is differentiable and $|\partial_\psi \psi_s|/(1+\psi_s)$ is locally uniformly bounded over observation dates. The corresponding sufficient conditions are $v_{t+j,\psi}^{\tau+j} \le 0$ for every $j \ge 0$ and $\partial_\psi \Delta_{t+j}^V \ge 0$ for every $j \ge 1$. If one variance derivative is strictly negative at a finite future date, the product-drop derivative is strictly positive; adding $\partial_\psi \Delta_t^Y \ge 0$ makes the search derivative strictly positive. The derivative includes all historical and future processing changes specified by the family.

The opportunity restrictions (OA.A.83) concern solved value functions; they are not primitive conditions. In particular, higher values of D_t and D_{t+1} separately do not order their difference $x\Delta_t^Y = D_t - dD_{t+1}$. The results below give primitive innovation restrictions and stationary specializations in which opportunity ordering also holds.

Proof. At a common argument U, the maximum (OA.A.81) is nonincreasing in Δ^V. The innovation ordering is a convex-order comparison of mean-one lognormal demand transitions. Backward induction in the recursion for U, with the actual opportunity paths fixed and common terminal retained-product values, therefore gives $U_t^H(\mathcal{B},\tau) \le U_t^L(\mathcal{B},\tau)$. The bound on expected discounted payoffs in (OA.A.28) passes this inequality to the infinite horizon. If an innovation inequality is strict at a finite future date, Gaussian full support gives positive probability of retaining the existing product until that date and reaching a strictly convex active-production region. The inequality in U_t is then strict at every $\mathcal{B} > 0$. The product-drop matching level is zero in both economies; the

additional current opportunity ordering gives $U_t^H - \Delta_t^{V,H} \leq U_t^L - \Delta_t^{V,L}$ for the search comparison. The root orderings follow.

For the local statement, put $G_j = \partial_\psi U_{t+j}$ along the retained-product history. Let p equal 1, $1-x$, or 0 under no search, continue-and-search, or drop-and-search, respectively. Differentiating the Gaussian expectation and the optimal continuation action yields

$$G_j = d\Gamma_j v_{t+j,\psi}^{\tau+j} + d\mathbb{E}\left[p_{j+1}G_{j+1} - x\mathbf{1}_{\{ns\}}\partial_\psi \Delta_{t+j+1}^V\right], \qquad \Gamma_j > 0. \tag{OA.A.84}$$

Here Γ_j is the marginal increase in the expected continuation maximum from increasing its innovation variance, holding the next values fixed. Discounted iteration makes $G_0 < 0$ under the stated strictness condition. Implicit differentiation gives $\partial_\psi \mathcal{B}_{it}^{ds} = -G_0/U_{t,\mathcal{B}} > 0$ and $\partial_\psi \mathcal{B}_{it}^{cs} = (\partial_\psi \Delta_t^V - G_0)/U_{t,\mathcal{B}} > 0$ when its additional condition holds.

Local statements require the usual discounted differentiation of the specified path family. One sufficient regularity condition is a locally uniform bound on $|\partial_\psi \psi_s|/(1+\psi_s)$ over its observation dates. It bounds $|\partial_\psi v_s|/v_s$ uniformly; cumulative innovation variance along one product is bounded by σ_0^2. Writing each normal innovation as its standard deviation times a standard normal variable bounds the expected absolute derivative of each potential product's gross operating payoffs uniformly in tenure. There are at most $j+1$ potential products by future date j, so the expected absolute payoff derivatives are bounded by a constant times the convergent sum $\sum_{j\geq 0}(j+1)d^j$. Future Gaussian states hit switching boundaries with probability zero, permitting the action-envelope differentiation above. This condition holds for proportional path changes at a positive processing level. For the constant-processing and future-plateau specializations below, the bound also holds at zero processing and permits right derivatives there. □

Precision conditions for a general processing path. For a differentiable comparison indexed by λ, put $q_j = 1 + \psi_{i,t+j}$ for $j \geq 1$ and $k_j = k_0 + \sum_{\ell=1}^{j} q_\ell$, where k_0 is inherited precision as defined in (OA.A.60). Then $v_j = \sigma_\varepsilon^2 q_{j+1}/[k_j(k_j + q_{j+1})]$, so the primitive conditions for innovation contraction are

$$k_j^2 q'_{j+1} \leq q_{j+1}k'_j(2k_j + q_{j+1}), \qquad j \geq 0. \tag{OA.A.85}$$

For the strict threshold comparisons above, require a strict contraction at some finite date and the respective replacement-opportunity ordering. The latter concerns optimal values: better future information does not by itself order $\Delta_t^V = (D_t - dD_{t+1})/x$. Stationary future processing establishes

this ordering automatically; a general age-dependent path requires the additional value conditions in Proposition OA.3. The following result reduces the innovation restrictions to one primitive inequality for a proportional, concave profile.

Corollary OA.3 (A primitive restriction on existing-product innovations)**.** *For a proportional processing comparison, let $\psi_{is} = \psi h(a_{is})/h(a_{it})$ with the profile h fixed. Assume h is positive at observation ages and has a nonnegative, nondecreasing, concave extension to age zero; observations in the existing-product history begin at age at least one. Define*

$$c = \frac{\sigma_\varepsilon^2}{\sigma_0^2}, \qquad \omega = \frac{\sum_{s=t-\tau+1}^{t} h(a_{is})}{h(a_{i,t+1})}, \qquad Q = 1 + \psi \frac{h(a_{i,t+1})}{h(a_{it})}.$$

The single primitive condition

$$Q\sqrt{\omega(\omega+1)} \geq c + \tau - \omega \tag{OA.A.86}$$

makes the current innovation derivative weakly negative and every subsequent retained-product innovation derivative strictly negative. For a finite comparison, replace Q by $\sqrt{Q_H Q_L}$ to obtain the same variance ordering. Together with the respective opportunity conditions in Proposition OA.3, this gives its strict threshold conclusions. The opportunity ordering remains an additional condition on values.

Proof. Put $C_* = c + \tau - \omega > 0$. The current innovation variance is $\sigma_\varepsilon^2 Q/[(C_* + \omega Q)(C_* + (\omega+1)Q)]$. Its derivative has the sign of $C_*^2 - \omega(\omega+1)Q^2$; the finite-pair difference has the sign of $C_*^2 - \omega(\omega+1)Q_H Q_L$.

At future retained tenure $n = \tau + j$, put $h_+ = h(a_{i,t+j+1})$ and $e = h(a_{i,t+j+2}) - h_+$. Let H_j sum h over the existing product's n observations. Concavity and the age-zero extension give $h_+ \geq (n+1)e$ and $H_j \leq nh_+ - en(n+1)/2$. Consequently

$$h_+^2 - e(2H_j + h_+) \geq (h_+ - ne)(h_+ - (n+1)e) \geq 0, \qquad \omega_{j+1} - \omega_j \geq \tfrac{1}{2}.$$

If $e = 0$, the ratio increases by one. Since the derivative of $\sqrt{\omega(\omega+1)}$ exceeds one and $Q_j \geq 1$ is nondecreasing, $Q_j\sqrt{\omega_j(\omega_j+1)} + \omega_j - (\tau + j)$ strictly increases at each observation. Thus the initial inequality strengthens strictly at every subsequent retained date. The proof for finite pairs replaces Q_j by the nondecreasing $\sqrt{Q_{H,j}Q_{L,j}}$. At a positive processing level the proportional family also satisfies the preceding local regularity condition. □

The same propagation argument permits actual startup values below a nonnegative increasing

concave reference profile, provided they coincide with that reference at all ordinary ages. Use the actual historical sum: it is no greater than the reference sum, so both bounds on h_+ and H_j used in the proof remain valid. Without these profile or reference conditions, the full sequence of variance restrictions in Proposition OA.3 must be checked directly.

Corollary OA.4 (Constant processing)**.** *Suppose $\psi_{is} \equiv \psi$ at every date, and put $c = \sigma_\varepsilon^2/\sigma_0^2$. If*

$$(1+\psi)\sqrt{\tau(\tau+1)} \geq c, \tag{OA.A.87}$$

both processing derivatives are strictly positive. For a finite comparison $\psi_H > \psi_L$, the endpoint condition $\sqrt{(1+\psi_H)(1+\psi_L)\tau(\tau+1)} \geq c$ gives strict ordering of both thresholds.

Proof. Put $q = 1+\psi$. At retained tenure n, the innovation variance and its derivative are

$$\begin{aligned} v_n &= \frac{\sigma_\varepsilon^2 q}{(c+nq)[c+(n+1)q]}, \\ \partial_\psi v_n &= \frac{\sigma_\varepsilon^2[c^2 - n(n+1)q^2]}{(c+nq)^2[c+(n+1)q]^2}. \end{aligned} \tag{OA.A.88}$$

Condition (OA.A.87) contracts the current innovation weakly and every later innovation strictly. The finite-pair difference has the sign of $c^2 - n(n+1)q_H q_L$, giving the stated endpoint condition.

Lemma OA.4 gives $\Delta_\psi^V > 0$ under stationary future processing, with a right derivative at zero. Proposition OA.3 therefore applies, including equality in the current stock inequality because later innovation comparisons are strict. Equivalently, the local condition is (OA.A.79) with $k = c + \tau q$, $k' = \tau$, and $q' = 1$. □

Corollary OA.5 (A stationary future with a varying processing history)**.** *At the current date write $\psi_{is} = \psi r_s$, with $r_t = 1$, $0 \leq r_s \leq R$, and $r_s = R > 0$ for every $s \geq t+1$. Thus ψ is current processing and ψR is the future plateau; historical observations retain their actual precisions. The comparison holds r_s and R fixed, so the future level ψR rises with ψ; it is distinct from the main-text comparison that holds future processing fixed. Put $c = \sigma_\varepsilon^2/\sigma_0^2$ and*

$$\omega = \frac{\sum_{s=t-\tau+1}^{t} r_s}{R}, \qquad Q = 1+\psi R, \qquad C_* = c + \tau - \omega.$$

If $Q\sqrt{\omega(\omega+1)} \geq C_*$, both processing derivatives are strictly positive, at any $d \in (0,1)$. For $\psi_H > \psi_L$, the condition $\sqrt{Q_H Q_L \omega(\omega+1)} \geq C_*$ gives strict finite ordering of both thresholds.

Proof. After j more observations of the existing product,

$$v_{t+j}^{\tau+j} = \frac{\sigma_\varepsilon^2 Q}{[C_* + (\omega + j)Q][C_* + (\omega + j + 1)Q]}.$$

Its processing derivative has the sign of $C_*^2 - (\omega + j)(\omega + j + 1)Q^2$. The finite-pair difference has the sign of $C_*^2 - (\omega + j)(\omega + j + 1)Q_H Q_L$. Each current weak inequality therefore becomes strict at all later retained dates. Every new product first produces at the common prior and receives its first observation next period, when the plateau already applies. Consequently $D_t = D_{t+1} = \cdots = D$ is the stationary vacancy value at processing ψR, and $\Delta^V = (1 - d)D/x$ strictly increases with ψ by Lemma OA.4. The general comparison proves the result. □

This restriction concerns the entire future from the next observation onward; a plateau reached only at a later date does not establish the result today. The constant-processing and future-plateau corollaries are supportive cases, not a claim that an arbitrary rising processing path orders replacement opportunities.

OA.A.12 A three-period model

This truncation separates inherited information, new signals, and replacement opportunities while preserving the model's production timing. There are decisions at dates 0 and 1, production at dates 1 and 2, and no continuation after date-2 production. At date 0, the existing product has tenure $\tau \geq 1$ and an updated belief, with two production opportunities remaining. A product retained at date 0 produces at date 1, its next signal informs the date-1 decision, and the selected product then produces at date 2. A new product starts from the common prior, as in the main model. All comparisons hold aggregate conditions and the prior fixed. Suppress the firm index, abbreviate $\mathcal{A}(z_i)$ by A and z_i by z, and write $\overline{\pi}(\mathcal{B}, z) = A\mathcal{B}^\eta - f^o$. Recall that $A_z > 0$, and let $d = \beta(1 - \delta)$, $\alpha = (\eta - 1)/\eta$, and $c = \sigma_\varepsilon^2/\sigma_0^2$. Maintain $0 < d, x < 1$, $\eta > 1$, positive costs and variances, and the terminal opportunity condition $\Delta_1^V = \beta\overline{\pi}(\mathcal{B}^0, z) - f^s/x > 0$.

At date 1, let F_1 denote the value above dropping and searching. The terminal problem gives

$$\begin{aligned} F_1(\mathcal{B}) &= (1 - x)[\beta\overline{\pi}(\mathcal{B}, z)]_+ + x[\beta\overline{\pi}(\mathcal{B}, z) - \Delta_1^V]_+, \\ \mathcal{B}_1^{\mathrm{ds}} &= \left(\frac{f^o}{A}\right)^{1/\eta}, \qquad \mathcal{B}_1^{\mathrm{cs}} = \left[(\mathcal{B}^0)^\eta - \frac{f^s}{x\beta A}\right]^{1/\eta}. \end{aligned} \tag{OA.A.89}$$

The two terminal demand thresholds are independent of processing and tenure; productivity lowers

the first and raises the second. To study learning before the last decision, define the explicit Gaussian expectation

$$H(\mathcal{B}, v) = \mathbb{E}F_1\left(\mathcal{B}\exp\{\alpha\sqrt{v}Z - \alpha^2 v/2\}\right), \qquad Z \sim N(0,1). \tag{OA.A.90}$$

Under constant processing $q = 1 + \psi$, the next innovation variance is

$$\begin{aligned} v_\tau &= \frac{\sigma_\varepsilon^2 q}{(c+\tau q)[c+(\tau+1)q]}, \\ v_{\tau,\psi} &= \frac{\sigma_\varepsilon^2[c^2 - \tau(\tau+1)q^2]}{(c+\tau q)^2[c+(\tau+1)q]^2}, \qquad v_{0,\psi} = \frac{\sigma_\varepsilon^2}{(c+q)^2} > 0. \end{aligned} \tag{OA.A.91}$$

Subtracting dV_1^{ds} from V_0^{ns} gives the date-0 relative value and replacement opportunity:

$$\begin{aligned} U_0(\mathcal{B}, \tau) &= \beta\overline{\pi}(\mathcal{B}, z) + dH(\mathcal{B}, v_\tau), \qquad \Delta_0^V = \Delta_1^V + dH(\mathcal{B}^0, v_0), \\ U_0(\mathcal{B}_0^{\mathrm{ds}}, \tau) &= 0, \qquad U_0(\mathcal{B}_0^{\mathrm{cs}}, \tau) = \Delta_0^V. \end{aligned} \tag{OA.A.92}$$

In particular, the new product's learning gain remains part of the date-0 replacement opportunity.

Proposition OA.4 (Three-period comparative statics)**.** *Maintain the terminal opportunity condition above. For the date-0 thresholds, the following statements hold.*

1. *Under a common processing change at all dates,* $\operatorname{sgn}\mathcal{B}_{0,\psi}^{\mathrm{ds}} = \operatorname{sgn}\{(1+\psi)\sqrt{\tau(\tau+1)} - c\}$*. If* $(1+\psi)\sqrt{\tau(\tau+1)} \geq c$*, then* $\mathcal{B}_{0,\psi}^{\mathrm{cs}} > 0$*; the drop derivative is zero at equality and positive above it. Below this bound, the drop derivative is negative, and the search sign is determined by* (OA.A.94) *below.*

2. *Both* $\mathcal{B}_{0,z}^{\mathrm{ds}} < 0$ *and* $\mathcal{B}_{0,z}^{\mathrm{cs}} > 0$ *hold under the primitive cost restriction*

$$\frac{d(1-x)}{1/x - d} < \frac{f^s}{\beta f^o} < 1 - x + \frac{2}{d}. \tag{OA.A.93}$$

   This interval is nonempty for every $d, x \in (0,1)$*. The separate primitive tests in* (OA.A.98) *below can establish either sign outside this interval.*

3. *At the same firm age and with the same future processing technology, an additional earlier observation raises both date-0 thresholds strictly. Under constant processing,* $\mathcal{B}_0^j(\tau+1) > \mathcal{B}_0^j(\tau)$ *for* $j \in \{\mathrm{ds}, \mathrm{cs}\}$*.*

Proof. The function F_1 is increasing and convex, with strict curvature on a set of positive Gaussian probability. Consequently $H_{\mathcal{B}} > 0$ and $H_v > 0$ for $\mathcal{B}, v > 0$, and $U_{0,\mathcal{B}} > 0$. For completeness, these derivatives can also be checked directly from the normal-CDF expression below. Differentiating the matching equations gives

$$\begin{aligned} \mathcal{B}^{\mathrm{ds}}_{0,\psi} &= -\frac{dH_v(\mathcal{B}^{\mathrm{ds}}_0, v_\tau)v_{\tau,\psi}}{U_{0,\mathcal{B}}(\mathcal{B}^{\mathrm{ds}}_0, \tau)}, \\ \mathcal{B}^{\mathrm{cs}}_{0,\psi} &= \frac{d\{H_v(\mathcal{B}^0, v_0)v_{0,\psi} - H_v(\mathcal{B}^{\mathrm{cs}}_0, v_\tau)v_{\tau,\psi}\}}{U_{0,\mathcal{B}}(\mathcal{B}^{\mathrm{cs}}_0, \tau)}. \end{aligned} \tag{OA.A.94}$$

Equation (OA.A.91) proves part (i). The second identity is an exact comparison at an endogenous threshold, rather than an additional primitive sufficient condition. It permits a positive search derivative even when the next signal raises the continuation value of the existing product. For part (iii), an additional earlier observation increases inherited precision and strictly lowers v_τ, while leaving the new-product problem and Δ^V_0 unchanged. The two matching values in (OA.A.92) are therefore unchanged, whereas U_0 falls strictly at each positive demand state; both roots rise.

For part (ii), write $r = f^s/(\beta f^o)$, $K_d = f^o/A$, $K_c = (\mathcal{B}^0)^\eta - f^s/(x\beta A) > K_d$, and $\mathcal{B}_* = K_d^{1/\eta}$. Define the following probabilities, all evaluated at primitive states:

$$\begin{aligned} p_j(\mathcal{B}, v) &= \Phi\left(\frac{\log(\mathcal{B}^\eta/K_j) - \eta\alpha^2 v/2}{\eta\alpha\sqrt{v}}\right), \qquad j \in \{d, c\}, \\ p^0_j &= p_j(\mathcal{B}^0, v_0), \qquad p^*_c = p_c(\mathcal{B}_*, v_\tau) < \frac{1}{2}. \end{aligned} \tag{OA.A.95}$$

The terminal envelope and (OA.A.92) imply

$$AU_{0,A}(\mathcal{B}, \tau) = U_0(\mathcal{B}, \tau) + \beta f^o + d\{(1-x)\beta f^o p_d(\mathcal{B}, v_\tau) - f^s p_c(\mathcal{B}, v_\tau)\}. \tag{OA.A.96}$$

Differentiating the terminal action payoffs yields this identity; switching ties have probability zero. At the drop threshold, $U_0 = 0$ and $\mathcal{B}^{\mathrm{ds}}_0 < \mathcal{B}_*$ because $H > 0$. Since $p_d \geq p_c$, equation (OA.A.96) gives

$$AU_{0,A}(\mathcal{B}^{\mathrm{ds}}_0, \tau) \geq \beta f^o\{1 - d[r - (1-x)]_+ p^*_c\}.$$

At the search threshold, $U_0(\mathcal{B}^0, 0) - U_0(\mathcal{B}^{\mathrm{cs}}_0, \tau) = f^s/x$; subtracting (OA.A.96) between these states and using $0 \leq p_j \leq 1$ gives

$$A\{U_{0,A}(\mathcal{B}^0, 0) - U_{0,A}(\mathcal{B}^{\mathrm{cs}}_0, \tau)\} \geq \beta f^o\{r(1/x - dp^0_c) - d(1-x)(1-p^0_d)\}. \tag{OA.A.97}$$

Thus the separate primitive sufficient conditions are

$$d[r-(1-x)]_+ p_c^* < 1 \implies \mathcal{B}_{0,z}^{\mathrm{ds}} < 0, \qquad r > \frac{d(1-x)(1-p_d^0)}{1/x - dp_c^0} \implies \mathcal{B}_{0,z}^{\mathrm{cs}} > 0. \tag{OA.A.98}$$

The denominator on the right is positive. Replacing p_c^* by $1/2$ and the prior probabilities by their worst-case bounds yields (OA.A.93). Its lower endpoint is below $x < 1$, while its upper endpoint exceeds two, so the interval is nonempty. Finally, implicit differentiation of (OA.A.92), using $A_z > 0$ and $U_{0,\mathcal{B}} > 0$, establishes the stated productivity signs. □

Evaluation from primitives. The expectation in (OA.A.90) is explicit. Let $\chi = \eta(\eta-1)\alpha^2/2$, $M = \mathcal{B}^\eta e^{\chi v}$, $\ell_2 = [\log(\mathcal{B}^\eta/K) - \eta\alpha^2 v/2]/(\eta\alpha\sqrt{v})$, and $\ell_1 = \ell_2 + \eta\alpha\sqrt{v}$. Then, with Φ and ϕ denoting the standard normal CDF and density,

$$\begin{aligned} \mathcal{C}(\mathcal{B}, K, v) &= M\Phi(\ell_1) - K\Phi(\ell_2), \\ \mathcal{C}_v &= \chi M\Phi(\ell_1) + \frac{\eta\alpha M}{2\sqrt{v}}\phi(\ell_1) > 0, \\ H(\mathcal{B}, v) &= \beta A\{(1-x)\mathcal{C}(\mathcal{B}, K_d, v) + x\mathcal{C}(\mathcal{B}, K_c, v)\}. \end{aligned} \tag{OA.A.99}$$

These formulas make the exact search comparison computable without solving a further dynamic problem. They do not make an inequality evaluated at $\mathcal{B}_0^{\mathrm{cs}}$ a primitive condition.

Inherited information versus a common processing change. The same calculation allows a non-constant history. Let p be inherited posterior precision and let q_+ be the precision of the next signal; these quantities are precisions, whereas $q = 1 + \psi$ in (OA.A.91) is a processing multiplier. For a profile parameter ξ,

$$v = p^{-1} - (p+q_+)^{-1}, \qquad v_\xi = \frac{p^2 q_{+,\xi} - q_+(2p+q_+)p_\xi}{p^2(p+q_+)^2}. \tag{OA.A.100}$$

If only inherited information improves, holding future processing fixed, then $p_\xi > 0$, $q_{+,\xi} = 0$, and the replacement opportunity is unchanged; both thresholds rise. If processing increases throughout the history and the next observation, both precisions change: inherited information reduces the value of further learning, whereas a more precise next signal increases it. The drop derivative has sign $-v_\xi$. If $v_\xi \le 0$ and $q_{+,\xi} > 0$, the search threshold rises because the new product also benefits from the more precise next signal. The constant-processing turning point in part (i) is the exact

balance of these two effects, rather than an assumption that future processing is held fixed.

Under part (ii), productivity widens the gap $\mathcal{B}_0^{\mathrm{cs}} - \mathcal{B}_0^{\mathrm{ds}}$. Whenever the exact search comparison in (OA.A.94) is positive below the processing turning point, processing also widens this gap because the drop threshold falls. An increase in both thresholds alone does not determine the change in their gap.

OA.A.13 Returns to Pre-Entry Investment

Proposition OA.5 (Returns to Pre-Entry Investment)**.** *Hold aggregate prices, quantities, and the joint distribution of normalized types $(\varphi_i, \widetilde{m}_i)$ fixed, with $z_i = z_0\varphi_i$ and $m_i = m_0\widetilde{m}_i$. Let $\mathcal{E}(m_0, z_0) = \mathbb{E}_\varphi[V^e(\varphi)]$ denote expected entry value before investment costs. Under optimal policies, write $\mathcal{E} = \mathcal{G} - \mathcal{K}$, where $\mathcal{G}$ is expected discounted revenue less variable production costs and $\mathcal{K}$ is expected discounted operating and search fixed costs, valued at fixed wages. Both include replacements and the corresponding components of assigned vacant-line payoffs at endogenous exit.*[1]

Expected entry value is weakly increasing in m_0. Where the derivatives exist and differentiation under expectations is valid,

$$\frac{\partial \mathcal{E}}{\partial z_0} = \frac{\eta - 1}{z_0}\mathcal{G}, \tag{OA.A.101}$$

$$\frac{\partial^2 \mathcal{E}}{\partial z_0\, \partial m_0} = \frac{\eta - 1}{z_0}\frac{\partial \mathcal{G}}{\partial m_0} = \frac{\eta - 1}{z_0}\left(\frac{\partial \mathcal{E}}{\partial m_0} + \frac{\partial \mathcal{K}}{\partial m_0}\right). \tag{OA.A.102}$$

The productivity derivative is strictly positive if production occurs with positive probability. Local complementarity holds if and only if $\partial \mathcal{G}/\partial m_0 \geq 0$. A sufficient condition for strict complementarity is

$$\frac{\partial \mathcal{E}}{\partial m_0} > 0, \qquad \frac{\partial \mathcal{K}}{\partial m_0} \geq 0. \tag{OA.A.103}$$

Proof. For fixed normalized types, higher m_0 weakly increases processing at every age. Adding independent normal noise to the additional signals reproduces the lower-capability information structure, jointly with market observations. The firm can therefore follow the same production and product decisions, including the continuation used to evaluate assigned exit payoffs. Since managerial ability is observed before the first update, averaging over types and optimizing startup operation establishes managerial monotonicity.

[1] Assigned exit payoffs cover the initial line at age one and all lines thereafter; preproduction abandonment contributes zero.

Fix policies for startup operation, subsequent product decisions, and assigned exit continuations as functions of observed histories, optimizing quantities conditional on those histories. The history distribution is independent of z_0 under the fixed type distribution and information-processing path. By (2.10), policy value equals $z_0^{\eta-1}H - \mathcal{K}$, with H and $\mathcal{K}$ independent of z_0. The envelope theorem gives (OA.A.101), with $\mathcal{G} = z_0^{\eta-1}H$ evaluated at the optimal policy. Boundary terms cancel because competing net payoffs coincide at indifference, including startup abandonment and age-one exit.

On a compact interval of positive z_0, each fixed policy's productivity derivative is bounded by a constant times its gross operating payoffs. Integrability over types and locally uniform derivative bounds justify the infinite-horizon limit and differentiation under expectations, using finite-horizon approximations and the discounted-payoff bound above (Stokey and Lucas Jr, 1989, Sections 9.1 and 9.4). Production with positive probability implies $\mathcal{G} > 0$.

Differentiating (OA.A.101) with respect to m_0 and using $\mathcal{G} = \mathcal{E} + \mathcal{K}$ gives (OA.A.102) and (OA.A.103). The derivatives of $\mathcal{G}$ and $\mathcal{K}$ include changes in optimal decisions; only their difference satisfies the envelope cancellation. □

Holding inherited capabilities fixed, the capability laws imply that every differentiable interior optimum of (2.24) satisfies

$$(1-\zeta_m)m_{0t}\mathcal{E}_{m_{0t}} = w_t\chi\alpha_m^e(\iota_t^m)^\chi, \qquad (1-\zeta_z)z_{0t}\mathcal{E}_{z_{0t}} = w_t\chi\alpha_z^e(\iota_t^z)^\chi.$$

Stationary consistency is imposed after private optimization.

OA.B Model Extension

OA.B.1 Persistent Product-Market Shocks

This appendix collects the persistent-shock equations under the baseline timing, entry, and exit conventions. Calendar-time subscripts are suppressed only for stationary aggregates.

Demand and information. With the prior $\log\theta_{ip} \sim N(\theta_0, \sigma_0^2)$, realized demand satisfies

$$\log\theta_{ipt} = \log\theta_{ip} + \varepsilon_{ipt}, \tag{OA.B.1}$$

$$\varepsilon_{ip,t+1} \mid \varepsilon_{ipt} \sim N\left(\rho\varepsilon_{ipt}, (1-\rho^2)\sigma_\varepsilon^2\right), \qquad 0 \le \rho < 1. \tag{OA.B.2}$$

Market innovations are independent across periods and lines and independent of expertise and private-signal errors. New products start with stationary shocks $\varepsilon \sim N(0, \sigma_\varepsilon^2)$. The additional signal remains

$$\log s_{ipt} = \log \theta_{ip} + \upsilon_{ipt}, \qquad \upsilon_{ipt} \sim N\left(0, \frac{\sigma_\varepsilon^2}{\psi_{it}}\right). \tag{OA.B.3}$$

Signal errors retain their baseline independence assumptions. Beliefs condition on $\mathcal{I}_{ipt} = \{\theta_{ips}, s_{ips}\}_{s=t-\tau+1}^{t}$, where $\tau \equiv \tau_{ipt}$.

After the first observation, market outcomes enter through

$$\log \theta_{ip,t+1} - \rho \log \theta_{ipt} = (1-\rho)\log \theta_{ip} + \varepsilon_{ip,t+1} - \rho \varepsilon_{ipt}. \tag{OA.B.4}$$

The innovation has variance $(1-\rho^2)\sigma_\varepsilon^2$. Market precision is $1/\sigma_\varepsilon^2$ for the first observation and $(1-\rho)/[(1+\rho)\sigma_\varepsilon^2]$ thereafter.

Posterior beliefs. For $\tau \geq 1$, posterior moments are

$$\hat{\theta}_{ipt} = \hat{\sigma}_{ipt}^2 \left[\frac{\theta_0}{\sigma_0^2} + \frac{\log \theta_{ip,t-\tau+1}}{\sigma_\varepsilon^2} + \sum_{s=t-\tau+2}^{t} \frac{\log \theta_{ips} - \rho \log \theta_{ip,s-1}}{(1+\rho)\sigma_\varepsilon^2} + \sum_{s=t-\tau+1}^{t} \frac{\psi_{is} \log s_{ips}}{\sigma_\varepsilon^2} \right], \tag{OA.B.5}$$

$$\hat{\sigma}_{ipt}^2 = \left[\frac{1}{\sigma_0^2} + \frac{1}{\sigma_\varepsilon^2} + (\tau - 1)\frac{1-\rho}{(1+\rho)\sigma_\varepsilon^2} + \sum_{s=t-\tau+1}^{t} \frac{\psi_{is}}{\sigma_\varepsilon^2} \right]^{-1}. \tag{OA.B.6}$$

The first sum is empty at $\tau = 1$. For continuing products with positive tenure,

$$\hat{\sigma}_{ip,t+1}^2 = \left[\frac{1}{\hat{\sigma}_{ipt}^2} + \frac{1-\rho}{(1+\rho)\sigma_\varepsilon^2} + \frac{\psi_{i,t+1}}{\sigma_\varepsilon^2} \right]^{-1}, \tag{OA.B.7}$$

$$\hat{\theta}_{ip,t+1} = \hat{\sigma}_{ip,t+1}^2 \left[\frac{\hat{\theta}_{ipt}}{\hat{\sigma}_{ipt}^2} + \frac{\log \theta_{ip,t+1} - \rho \log \theta_{ipt}}{(1+\rho)\sigma_\varepsilon^2} + \frac{\psi_{i,t+1} \log s_{ip,t+1}}{\sigma_\varepsilon^2} \right]. \tag{OA.B.8}$$

From the common prior (θ_0, σ_0^2), the first update is

$$\hat{\sigma}_{ip,t+1}^2 = \left[\frac{1}{\sigma_0^2} + \frac{1+\psi_{i,t+1}}{\sigma_\varepsilon^2} \right]^{-1}, \tag{OA.B.9}$$

$$\hat{\theta}_{ip,t+1} = \hat{\sigma}_{ip,t+1}^2 \left[\frac{\theta_0}{\sigma_0^2} + \frac{\log \theta_{ip,t+1}}{\sigma_\varepsilon^2} + \frac{\psi_{i,t+1} \log s_{ip,t+1}}{\sigma_\varepsilon^2} \right]. \tag{OA.B.10}$$

These initial updates apply to entrants' and replacement products, using signal precision at the observation date.

State and joint transition. A positive-tenure line has state

$$\xi_{ipt} = (\hat{\theta}_{ipt}, \log \theta_{ipt}, \tau_{ipt}; a_{it}, \psi_{it}, \varphi_i, m_i). \tag{OA.B.11}$$

Posterior variance is determined by tenure and the information history. Current realized demand forecasts the persistent shock; the equivalent numerical coordinates are $(\hat{\theta}_{ipt}, \log \theta_{ipt} - \hat{\theta}_{ipt})$.

Conditional on retention,

$$\log \theta_{ip,t+1} \mid \mathcal{I}_{ipt} \sim N\Big(\rho \log \theta_{ipt} + (1-\rho)\hat{\theta}_{ipt}, (1-\rho)^2 \hat{\sigma}^2_{ipt} + (1-\rho^2)\sigma^2_\varepsilon\Big). \tag{OA.B.12}$$

The pair $(\hat{\theta}_{ip,t+1}, \log \theta_{ip,t+1})$ is jointly Gaussian, conditional on current information and firm states, with moments

$$\mathbb{E}_{it}\begin{pmatrix} \hat{\theta}_{ip,t+1} \\ \log \theta_{ip,t+1} \end{pmatrix} = \begin{pmatrix} \hat{\theta}_{ipt} \\ \rho \log \theta_{ipt} + (1-\rho)\hat{\theta}_{ipt} \end{pmatrix}, \tag{OA.B.13}$$

$$\mathrm{Var}_{it}\begin{pmatrix} \hat{\theta}_{ip,t+1} \\ \log \theta_{ip,t+1} \end{pmatrix} = \begin{pmatrix} \hat{\sigma}^2_{ipt} - \hat{\sigma}^2_{ip,t+1} & (1-\rho)\hat{\sigma}^2_{ipt} \\ (1-\rho)\hat{\sigma}^2_{ipt} & (1-\rho)^2\hat{\sigma}^2_{ipt} + (1-\rho^2)\sigma^2_\varepsilon \end{pmatrix}. \tag{OA.B.14}$$

The cross-covariance uses

$$\mathrm{Cov}_{it}(\hat{\theta}_{ip,t+1}, \log \theta_{ip,t+1}) = \mathrm{Cov}_{it}(\log \theta_{ip}, \log \theta_{ip,t+1}).$$

For a tenure-zero product, the initial pair is jointly Gaussian with moments

$$\mathbb{E}_{it}\begin{pmatrix} \hat{\theta}_{ip,t+1} \\ \log \theta_{ip,t+1} \end{pmatrix} = \begin{pmatrix} \theta_0 \\ \theta_0 \end{pmatrix}, \tag{OA.B.15}$$

$$\mathrm{Var}_{it}\begin{pmatrix} \hat{\theta}_{ip,t+1} \\ \log \theta_{ip,t+1} \end{pmatrix} = \begin{pmatrix} \sigma^2_0 - \hat{\sigma}^2_{ip,t+1} & \sigma^2_0 \\ \sigma^2_0 & \sigma^2_0 + \sigma^2_\varepsilon \end{pmatrix}, \tag{OA.B.16}$$

where (OA.B.9) gives the next posterior variance. Let $g_{it}(d\xi' \mid \xi)$ include these Gaussian laws and $a_{i,t+1} = a_{it} + 1$, $\tau_{ip,t+1} = \tau_{ipt} + 1$. Write $\xi^0(a; \varphi, m)$ for the common-prior state, without realized demand, and $\mathbb{E}^0_{it}$ for its initial transition.

Production. The demand term for positive tenure is

$$\begin{aligned}\mathcal{B}_{it}(\xi) &\equiv \mathbb{E}_{it}\left[\theta_{ip,t+1}^{(\eta-1)/\eta}\right], \\ \log \mathcal{B}_{it}(\xi) &= \frac{\eta-1}{\eta}\left[\rho \log \theta_{ipt} + (1-\rho)\hat{\theta}_{ipt}\right] \\ &\quad + \frac{1}{2}\left(\frac{\eta-1}{\eta}\right)^2\left[(1-\rho)^2\hat{\sigma}_{ipt}^2 + (1-\rho^2)\sigma_\varepsilon^2\right]. \end{aligned} \tag{OA.B.17}$$

For a tenure-zero product,

$$\mathcal{B}^0 = \exp\left\{\frac{\eta-1}{\eta}\theta_0 + \frac{1}{2}\left(\frac{\eta-1}{\eta}\right)^2(\sigma_0^2 + \sigma_\varepsilon^2)\right\}. \tag{OA.B.18}$$

Quantity, production labor, realized price, and expected profit satisfy

$$y_{ip,t+1}(\xi) = \left(\frac{\eta-1}{\eta}\right)^\eta P_{p,t+1}^\eta C_{p,t+1}\mathcal{B}_{it}(\xi)^\eta\left(\frac{z_i}{w_{t+1}}\right)^\eta, \tag{OA.B.19}$$

$$\ell_{ip,t+1}(\xi) = \left(\frac{\eta-1}{\eta}\right)^\eta P_{p,t+1}^\eta C_{p,t+1}\mathcal{B}_{it}(\xi)^\eta z_i^{\eta-1} w_{t+1}^{-\eta}, \tag{OA.B.20}$$

$$p_{ip,t+1} = \frac{\eta}{\eta-1}\frac{w_{t+1}}{z_i}\frac{\theta_{ip,t+1}^{(\eta-1)/\eta}}{\mathcal{B}_{it}(\xi)}, \tag{OA.B.21}$$

$$\overline{\pi}_{it+1}(\xi) = \frac{(\eta-1)^{\eta-1}}{\eta^\eta}\left(\frac{z_i}{w_{t+1}}\right)^{\eta-1} P_{p,t+1}^\eta C_{p,t+1}\mathcal{B}_{it}(\xi)^\eta - w_{t+1}f^o. \tag{OA.B.22}$$

At tenure zero, use $\mathcal{B}^0$. Expected price remains $\eta w_{t+1}/[(\eta-1)z_i]$.

Product-line values. Under the baseline timing and exit payoffs, the action values satisfy

$$V_{it}^{ns}(\xi) = \beta\overline{\pi}_{it+1}(\xi) + \beta(1-\delta)\mathbb{E}_{it}\left[V_{i,t+1}(\xi') \mid \xi\right], \tag{OA.B.23}$$

$$V_{it}^{cs}(\xi) = -w_t f^s + (1-x)V_{it}^{ns}(\xi) + xV_{it}^{ns}(\xi^0), \tag{OA.B.24}$$

$$V_{it}^{ds}(\emptyset) = -w_t f^s + xV_{it}^{ns}(\xi^0) + (1-x)\beta(1-\delta)V_{i,t+1}^{ds}(\emptyset). \tag{OA.B.25}$$

Common firm states are suppressed. Here ξ^0 is the fresh-product state at the current firm age, evaluated using $\mathcal{B}^0$ and the initial transition.

For firms aged two and older,

$$V_{it}(\xi) = \max\{V_{it}^{ns}(\xi), V_{it}^{cs}(\xi), V_{it}^{ds}(\emptyset)\}. \tag{OA.B.26}$$

Let $\mathcal{R}_{it}^{ns}$, $\mathcal{R}_{it}^{cs}$, and $\mathcal{R}_{it}^{ds}$ denote the maximizing regions. Where ordered cutoffs exist, they depend on $(\log\theta_{ipt}, \tau_{ipt}; a_{it}, \psi_{it}, \varphi_i, m_i)$ and satisfy

$$V_{it}^{cs}(\hat{\theta}^{ds}, \log\theta_{ipt}, \tau_{ipt}) = V_{it}^{ds}(\emptyset), \tag{OA.B.27}$$

$$V_{it}^{ns}(\hat{\theta}^{cs}, \log\theta_{ipt}, \tau_{ipt}) = V_{it}^{cs}(\hat{\theta}^{cs}, \log\theta_{ipt}, \tau_{ipt}). \tag{OA.B.28}$$

Equal demand terms need not imply equal continuation values, so both continuous state coordinates are retained.

Incumbent firm value is

$$V(\Theta_{it}; a_{it}, \psi_{it}, \varphi_i, m_i) = \sum_{k=1}^{n_i} V_{it}(\xi_{ik,t}), \qquad a_{it} \geq 2, \tag{OA.B.29}$$

with $V_{it}(\xi_{ik,t}) = V_{it}^{ds}(\emptyset)$ for vacant lines.

Startups and pre-entry decisions. Upon endogenous exit at age one, the initial line receives $V_{t+1}^{ds}(\emptyset)$, while the initially vacant lines receive zero. Its first retention decision has value

$$\begin{aligned} V_{t+1}^{e,1}(\hat{\theta}, \log\theta; \psi, \varphi, m) = \max\Big\{ & V_{t+1}^{ns}(\hat{\theta}, \log\theta, 1) + (n(\varphi)-1)V_{t+1}^{ds}(\emptyset), \\ & V_{t+1}^{cs}(\hat{\theta}, \log\theta, 1) + (n(\varphi)-1)V_{t+1}^{ds}(\emptyset),\ V_{t+1}^{ds}(\emptyset)\Big\}. \end{aligned} \tag{OA.B.30}$$

Firm states are $(1, \psi, \varphi, m)$, with maximizing regions $\mathcal{R}_{t+1}^{ns,e}$, $\mathcal{R}_{t+1}^{cs,e}$, and $\mathcal{R}_{t+1}^{ds,e}$. Retention requires $\max\{V^{ns}, V^{cs}\} + (n(\varphi)-1)V^{ds} \geq V^{ds}$; conditional on retention, the initial line searches if $V^{cs} > V^{ns}$. Exit precedes all search.

Before m is revealed, operating value conditional on φ is

$$\begin{aligned} \widetilde{V}_t^e(\varphi) = {} & \beta\overline{\pi}_{t+1}(\xi^0(0;\varphi)) \\ & + \beta(1-\delta)\mathbb{E}_{m|\varphi}\left[\mathbb{E}_t^0 V_{t+1}^{e,1}(\hat{\theta}', \log\theta'; \psi_{i,t+1}, \varphi, m)\right]. \end{aligned} \tag{OA.B.31}$$

Under the maintained single-crossing restriction in Section 2.4, the operating option and cutoff satisfy

$$V_t^e(\varphi) = \max\{\widetilde{V}_t^e(\varphi), 0\}, \qquad \underline{\varphi}_t^e = \inf\{\varphi : \widetilde{V}_t^e(\varphi) \geq 0\}. \tag{OA.B.32}$$

Zero-value matching holds at a continuous interior crossing. Pre-entry investments solve

$$(\iota_t^m, \iota_t^z) \in \arg\max_{\iota^m, \iota^z} \left\{ \mathbb{E}_\varphi[V_t^e(\varphi)] - w_t f_t^e(\iota^m, \iota^z) \right\}, \tag{OA.B.33}$$

using the baseline capability technologies and costs. Free entry requires

$$\mathbb{E}_\varphi[V_t^e(\varphi)] = w_t f_t^e(\iota_t^m, \iota_t^z). \tag{OA.B.34}$$

In stationarity, $M = m_0$, $Z = z_0$, $\iota^m = m_0$, and $\iota^z = z_0$.

Product-line measures. Let μ_{pt} record lines after search, with vacant lines indexed by their latest search destination. Initially vacant startup lines enter this measure after their first search. Let $\mathcal{X}_a$ include active states at age a, including the prior atom; $\mu_{pt}(\emptyset, a)$ is vacant-line mass. Firm states (ψ, φ, m) are suppressed and permanent types are integrated in aggregates.

The post-drop vacancy probability in symmetric equilibrium is

$$q_{t+1}^{\emptyset}(a') = \frac{\sum_q \left[\mu_{qt}(\emptyset, a'-1) + \int_{\mathcal{X}_{a'-1}} \int \mathbf{1}_{\mathcal{R}_{t+1}^{ds}}(\xi') g_t(d\xi' \mid \xi) \mu_{qt}(d\xi) \right]}{\sum_q \left[\mu_{qt}(\emptyset, a'-1) + \mu_{qt}(\mathcal{X}_{a'-1}) \right]}, \qquad a' \geq 2. \tag{OA.B.35}$$

Using the conditional product-of-marginals approximation in Appendix OA.A.5,

$$p_{t+1}^e(a') = \begin{cases} \left[q_{t+1}^{\emptyset}(a') \right]^{n(\varphi)-1}, & n(\varphi) > 1, \\ 1, & n(\varphi) = 1, \end{cases} \qquad a' \geq 2. \tag{OA.B.36}$$

Here p^e approximates the probability that all other lines are vacant, so dropped or vacant lines reach search with probability $1 - p^e$. Normalize (OA.B.35) only in age and type cells with positive mass.

For startups, endogenous-exit probability conditional on no exogenous death is exactly

$$p_{t+1}^e(1) = \int \mathbf{1}_{\mathcal{R}_{t+1}^{ds,e}}(\xi') g_t(d\xi' \mid \xi^0(0)). \tag{OA.B.37}$$

Each initially vacant line reaches search with probability $1 - p_{t+1}^e(1)$.

Continuing products and new assignments. For measurable $A \subseteq \mathcal{X}_{a+1}$ containing only positive-tenure states,

$$\mu_{p,t+1}(A) = (1-\delta)\int_{\mathcal{X}_a}\int_A \Big[\mathbf{1}_{\mathcal{R}^{ns}_{t+1}}(\xi') + (1-x)\mathbf{1}_{\mathcal{R}^{cs}_{t+1}}(\xi')\Big] g_t(d\xi' \mid \xi)\mu_{pt}(d\xi). \tag{OA.B.38}$$

At $a = 0$, use the startup regions $\mathcal{R}^{ns,e}_{t+1}$, $\mathcal{R}^{cs,e}_{t+1}$ and the initial transition.

Successful searches from incumbent source ages $a \geq 1$ generate

$$\begin{aligned}\mu_{p,t+1}(\xi^0(a+1)) = \frac{(1-\delta)x}{N-1}\sum_{q\neq p}\Bigg\{ \int_{\mathcal{X}_a}\int \Big[&\mathbf{1}_{\mathcal{R}^{cs}_{t+1}}(\xi') \\ &+ (1-p^e_{t+1}(a+1))\mathbf{1}_{\mathcal{R}^{ds}_{t+1}}(\xi')\Big] g_t(d\xi' \mid \xi)\mu_{qt}(d\xi) \\ &+ (1-p^e_{t+1}(a+1))\mu_{qt}(\emptyset, a)\Bigg\}.\end{aligned} \tag{OA.B.39}$$

For startups' first adjustment,

$$\begin{aligned}\mu_{p,t+1}(\xi^0(1)) = \frac{(1-\delta)x}{N-1}\sum_{q\neq p}\Bigg[\int \mathbf{1}_{\mathcal{R}^{cs,e}_{t+1}}(\xi') g_t(d\xi' \mid \xi^0(0)) \\ + (n(\varphi)-1)(1-p^e_{t+1}(1))\Bigg]\mu_{qt}(\xi^0(0)).\end{aligned} \tag{OA.B.40}$$

Under the baseline assignment mechanism and symmetry, destination probability $1/[N - n(\varphi)]$ over feasible markets averages to $1/(N-1)$ for each $p \neq q$.

The density of operating entrants is

$$\mu_{p,t+1}(\xi^0(0;\varphi,m)) = \frac{\mathcal{M}^e_{t+1}}{N}\frac{h_\varphi(\varphi)h_m(m \mid \varphi; m_{0,t+1})\mathbf{1}_{\{\varphi \geq \underline{\varphi}^e_{t+1}\}}}{1 - H_\varphi(\underline{\varphi}^e_{t+1})}. \tag{OA.B.41}$$

Here $\mathcal{M}^e_{t+1}$ is operating entrant mass. Without a single operating cutoff, replace the indicator by $\mathbf{1}_{\{\widetilde{V}^e_{t+1}(\varphi)\geq 0\}}$ and the denominator by its probability.

Vacant-line measures. For incumbent source ages $a \geq 1$,

$$\begin{aligned}\mu_{p,t+1}(\emptyset, a+1) = \frac{(1-\delta)(1-x)}{N-1}&\sum_{q\neq p}(1-p^e_{t+1}(a+1)) \\ &\times \left[\mu_{qt}(\emptyset,a) + \int_{\mathcal{X}_a}\int \mathbf{1}_{\mathcal{R}^{ds}_{t+1}}(\xi') g_t(d\xi' \mid \xi)\mu_{qt}(d\xi)\right].\end{aligned} \tag{OA.B.42}$$

Vacant lines are indexed by their failed-search destinations; failed replacement searches leave retained lines occupied.

For startups transitioning to age one,

$$\mu_{p,t+1}(\emptyset,1)=\frac{(1-\delta)(1-x)}{N-1}\sum_{q\neq p}(n(\varphi)-1)(1-p^e_{t+1}(1))\mu_{qt}(\xi^0(0)). \tag{OA.B.43}$$

Search resources and aggregation. Aggregate search labor is

$$\begin{aligned}\mathcal{F}^s_{t+1}=(1-\delta)f^s\sum_p\Bigg\{\sum_{a\geq 1}\Bigg[\int_{\mathcal{X}_a}\int\Big[\mathbf{1}_{\mathcal{R}^{cs}_{t+1}}(\xi')\\ +(1-p^e_{t+1}(a+1))\mathbf{1}_{\mathcal{R}^{ds}_{t+1}}(\xi')\Big]g_t(d\xi'\mid\xi)\mu_{pt}(d\xi)\\ +(1-p^e_{t+1}(a+1))\mu_{pt}(\emptyset,a)\Bigg]\\ +\mu_{pt}(\xi^0(0))\Bigg[\int\mathbf{1}_{\mathcal{R}^{cs,e}_{t+1}}(\xi')g_t(d\xi'\mid\xi^0(0))\\ +(n(\varphi)-1)(1-p^e_{t+1}(1))\Bigg]\Bigg\}.\end{aligned} \tag{OA.B.44}$$

The expression integrates over suppressed types and uses (OA.B.36); (2.34) gives exact firm-measure accounting.

Let $\mathcal{X}$ include all active states and prior atoms. Operating and entry resource costs are

$$\mathcal{F}^o_{t+1}=f^o\sum_p\mu_{pt}(\mathcal{X}),\qquad \mathcal{F}^e_{t+1}=f^e_{t+1}\frac{\mathcal{M}^e_{t+1}}{1-H_\varphi(\underline{\varphi}^e_{t+1})}. \tag{OA.B.45}$$

Entry costs apply to gross potential entrant mass, including types that abandon before production.

In stationary symmetric equilibrium, let μ denote the active measure in one product market, including permanent types. Then

$$L_p=\left(\frac{\eta-1}{\eta}\right)^\eta P^\eta C w^{-\eta}\int_{\mathcal{X}}\mathcal{B}(\xi)^\eta z(\varphi;z_0)^{\eta-1}\mu(d\xi), \tag{OA.B.46}$$

$$P=\frac{\eta}{\eta-1}w\left[\int_{\mathcal{X}}\mathcal{B}(\xi)^\eta z(\varphi;z_0)^{\eta-1}\mu(d\xi)\right]^{-1/(\eta-1)}, \tag{OA.B.47}$$

$$C=\frac{\overline{L}-\mathcal{F}^o-\mathcal{F}^s-\mathcal{F}^e}{N}\left[\int_{\mathcal{X}}\mathcal{B}(\xi)^\eta z(\varphi;z_0)^{\eta-1}\mu(d\xi)\right]^{1/(\eta-1)}. \tag{OA.B.48}$$

Labor-market clearing requires $\overline{L}=NL_p+\mathcal{F}^o+\mathcal{F}^s+\mathcal{F}^e$, and $\overline{C}=N^{\nu/(\nu-1)}C$. Normalize $w=1$.

The exact firm distribution follows the baseline portfolio transition on the expanded state space, with Γ^e normalized conditional on operating. The marginal laws above instead use (OA.B.36). Stationarity requires an invariant distribution, optimal decisions, free entry, and market clearing.

At $\rho = 0$, the model reduces to the baseline, and realized demand is no longer a necessary decision-state coordinate.

OA.B.2 Mapping Shipment Fixed Effects to Expertise

Under full information about permanent expertise, its contribution to $\log \mathcal{B}_{ip,t-1}$ has coefficient $(\eta - 1)/\eta$. Equation (2.11) therefore implies a total loading of $\eta - 1$ on $\log \theta_{ip}$ in log real shipments. After accounting for firm productivity and aggregate components, this gives the benchmark mapping $\alpha_{ip}^{PV} \simeq (\eta - 1) \log \theta_{ip}$. Demand-shock persistence affects shipment dynamics but does not change this full-information loading. We use this mapping as a benchmark approximation, with the prior mean anchored by the empirical fixed-effect normalization.

OA.C Additional Quantitative Results

OA.C.1 External Validation

This subsection compares untargeted model moments with published evidence on firm-size variation, demand and productivity over the firm life cycle, and within-firm sales concentration.

Sources of Firm-Size Variation. Table OA.1 compares the model's sources of firm-size variation with Hottman et al. (2016) (HRW). Demand-related components account for 64.44% of log deflated-sales variance, compared with a 75.80% contribution from firm appeal in HRW. Productivity and product scope jointly account for the remaining 35.56% in the model.

Demand and Productivity over the Firm Life Cycle. Table OA.2 compares demand and productivity gaps relative to old nonexiters with Foster et al. (2016) (FHS). The model reproduces entrants' larger disadvantage in demand than in productivity and the narrowing demand gap with age. It overstates the demand gaps, particularly among exiting firms, and generates a small entrant productivity disadvantage rather than the positive empirical coefficient.

Table OA.1: Sources of firm-size variation

Component	Contribution (%)
A. Model	
Productivity	26.67
Product scope	8.89
Product-specific expertise	6.58
Persistent demand	56.96
Imperfect learning	-0.37
Demand realization	1.27
Grouped contributions	
Productivity and product scope	35.56
Demand-related components	64.44
B. HRW	
Firm appeal	75.80
Product scope	21.03
Average marginal cost	-3.70
Cost dispersion	7.06
Markup	-0.20

Notes: Model contributions are each component's covariance with log deflated sales divided by the variance of log deflated sales. Demand-related components comprise product-specific expertise, persistent demand, imperfect learning, and demand realization. Productivity and potential scope both depend on φ_i; the grouped contributions summarize the components above. HRW values are from Table X, the all-firm cross-sectional decomposition. Their firm-appeal measure encompasses several model components, while the model has no markup heterogeneity. Values are rounded.

Table OA.2: Demand and productivity gaps relative to old nonexiters

	Demand		Productivity	
Group	Model	FHS	Model	FHS
Entrants (ages 0–4)	-1.39	-0.89	-0.05	0.02
Young (ages 5–9)	-0.27	-0.48	-0.03	0.01
Medium (ages 10–14)	-0.12	-0.38	-0.02	0.00
Exiters before next census	-2.19	-0.59	-0.08	-0.02

Notes: Entries are log-point coefficients relative to old nonexiters. Empirical estimates are from FHS Table 1. The model uses firm observations and date effects; FHS use plant observations and product–year effects. Model productivity is $\log z_i$; empirical productivity is estimated TFPQ. The demand measures likewise follow their respective model and empirical constructions. Age groups follow FHS.

Ex Ante Heterogeneity and Firm Growth. Using the employment-based statistical decomposition of Sterk et al. (2021) (SSP), Table OA.3 reports the share of log-employment variance attributable to ex ante heterogeneity. The model is close to the manufacturing estimate at age five, but its contribution remains high as firms age, whereas the empirical contribution declines.

Table OA.3: Ex ante shares of log-employment variance

Firm age	Model (%)	SSP manufacturing (%)
5	67.83	62.06
10	68.67	49.47
20	70.60	39.31

Notes: Both comparisons use balanced panels and SSP's statistical decomposition of employment. Manufacturing estimates are from SSP Appendix Table C.1, column (2). The model's age-20 value extrapolates beyond the age-0–19 estimation panel.

Table OA.4: Largest-product sales share by product scope

Product scope	Model (%)	Bernard et al. (%)
2	87.62	80.00
3	82.50	70.00
4	80.16	63.00
5	78.95	58.00
6	75.03	54.00
7	77.70	52.00
8	76.64	50.00
9	75.71	48.00
10	75.64	46.00

Notes: Entries are the mean sales share of the largest product across firms with the stated product scope. Products are ranked by sales within each firm. Empirical values are from Bernard et al. (2010), Table 10.

Within-Firm Sales Concentration. Table OA.4 compares the largest product's share of firm sales with Bernard et al. (2010). In both the model and data, the largest product generally accounts for a smaller share as product scope increases. The model nevertheless generates greater sales concentration, with larger differences among firms producing more products.

OA.C.2 Selection and Underlying Productive Potential

Firm Continuation and Size. Table OA.5 reports continuation and large-firm status by productivity and managerial ability. Among high-productivity firms, the joint probability of continuing and being large rises from 4.39% to 5.59% between the low- and high-managerial-ability groups. Thus, the higher large-firm share among continuing firms also translates into a higher probability relative to original entrants.

Current Output and Productive Potential. Table OA.6 compares high- and low-φ_i firms with comparable output at initial ages five and ten. We form 40 weighted output bins at each initial age and apply common bin weights to the two groups. Since φ_i determines both productivity and

Table OA.5: Continuation and large-firm status at age 20 (%)

Group	Continuation	Large | continuation	Continuing and large
Low z, low m	24.21	0.82	0.20
Low z, middle m	21.97	0.66	0.15
Low z, high m	15.77	1.46	0.23
Middle z, low m	6.02	1.22	0.07
Middle z, middle m	6.56	1.06	0.07
Middle z, high m	5.04	2.06	0.10
High z, low m	34.68	12.67	4.39
High z, middle m	35.40	14.09	4.99
High z, high m	30.07	18.61	5.59

Notes: Groups are defined by productivity z and managerial ability m and fixed at entry. Age-20 statistics pool firm-age masses over ages 18–22. Continuation and the joint probability are measured relative to original entrants; the conditional probability is measured among continuing firms. Large firms have employment at or above the age-specific 90th percentile among active firms, including ties.

Table OA.6: Current output and future outcomes by productive potential

Outcome	Initial age 5	Initial age 10
Current output gap (%)	4.50	1.72
Continuation gap (pp)	4.43	5.68
Active and in top output decile (pp)	10.14	12.37
Common-support coverage (%)	100.00	100.00

Notes: Comparisons are high minus low φ among firms active at each initial age, using common weights across 40 current-output bins and preserving ties. The current-output gap is based on standardized mean logs. Follow-up spans ten years. Continuation and top-decile status are measured at the end of the horizon. Top-decile status is based on output, whereas Table OA.5 uses employment. Managerial ability is not held fixed.

potential scope, the comparison captures their combined association with subsequent outcomes.

High-φ_i firms are more likely to be active and in the top output decile ten years later, with gaps of 10.14 and 12.37 percentage points at initial ages five and ten, respectively.

OA.C.3 Cohort Construction and Outcome Definitions

Figure 6 and Table 6 apply centered five-year moving averages to pooled weighted outcomes; age-five entries average ages 3–7.

For Table 6, we group original entrants by whether their initial product-specific log expertise is below or above θ_0, holding group membership fixed across paths. The three outcomes are measured before each period's exit decisions and are mutually exclusive and exhaustive. Denominators include all original entrants in each group, including subsequent exiters. The subsidy permanently raises managerial ability; the comparison describes its early-life effects.

Initial-product membership is recorded for firms with potential scope $n_i \leq 4$, covering 94.33% of original entrants. This restriction is imposed at entry in both paths and concerns potential capacity, not realized scope: all startups initially operate one product. The sales decomposition is not restricted by potential scope. Operating without the initial product means continuing with other products; it does not identify a particular replacement event. Initial-product retention reflects both product choices and firm exit.

OA.C.4 Aggregate Accounting and Resource Allocation

Accounting Decomposition. We decompose the equilibrium change in aggregate deflated sales into contributions from the firm distribution and within-group performance. Groups are defined by firm age, φ, and baseline managerial ability. Let μ_g^k and $\bar{r}_g^k$ denote firm mass and mean deflated sales per firm in group g, where $k = 0$ denotes the baseline and $k = 1$ the policy economy. For groups present in both economies,

$$\sum_g \left(\mu_g^1 \bar{r}_g^1 - \mu_g^0 \bar{r}_g^0\right) = \underbrace{\sum_g \frac{\bar{r}_g^0 + \bar{r}_g^1}{2}\left(\mu_g^1 - \mu_g^0\right)}_{\text{Firm distribution}} + \underbrace{\sum_g \frac{\mu_g^0 + \mu_g^1}{2}\left(\bar{r}_g^1 - \bar{r}_g^0\right)}_{\text{Within-group performance}}. \tag{OA.C.1}$$

Using averages of baseline and policy values allocates half of the interaction term to each component. We apply the same identity to separate firm mass from group shares within the distribution contribution, and employment per firm from deflated sales per worker within the performance contribution. These refinements preserve the two broad contributions. Within-group performance includes changes in the composition of firms within groups.

Table OA.7 reports the resulting contributions. Employment per firm and deflated sales per worker contribute 1.66 and 0.63 percentage points, respectively, accounting for the 2.29 percentage-point contribution from within-group performance in Table 8. With N fixed, consumption and aggregate deflated sales change proportionally, yielding identical contributions when normalized by their baseline values.

For the comparison holding firm mass fixed, we replace group masses with $\mathcal{M}^0 s_g^k$, where $\mathcal{M}^0$ is baseline firm mass and s_g^k is group g's firm share in economy k, and recompute the decomposition. This normalization measures changes in mean deflated sales per firm. Aggregate deflated sales rise by 8.90%, with composition and within-group performance contributing 6.51 and 2.38 percentage points, respectively. These contributions differ from those above because the normalization changes

Table OA.7: Decomposition of the equilibrium consumption change

Component	Contribution (pp)
Firm mass	-8.01
Firm composition	6.26
Employment per firm	1.66
Deflated sales per worker	0.63
Newly populated groups	0.00
Groups no longer populated	0.00
Total	0.54

Notes: Contributions are percentage points of baseline consumption. Firm mass and composition sum to the distribution contribution; employment per firm and deflated sales per worker sum to within-group performance. Groups present in only one economy are reported separately. Values are rounded.

Table OA.8: Labor allocation in the baseline and policy economies

Labor use	Baseline (%)	Policy (%)	Change (pp)
Production	89.54	89.48	-0.06
Operating costs	0.07	0.07	0.00
Product search	0.17	0.16	-0.01
Private entry	10.21	10.19	-0.02
Public program	0.00	0.11	0.11
Total	100.00	100.00	0.00

Notes: Shares are percentages of the fixed labor endowment. Firm employment includes production, operating, and search labor. Private entry resources include costs incurred by unsuccessful potential entrants. Program resources equal 0.1% of baseline consumption expenditure, corresponding to 0.11% of the labor endowment. Values are rounded.

the weights and interaction terms.

Labor Allocation. Table OA.8 reports how the fixed labor endowment is allocated across production, operating costs, search, entry, and the program. The program uses 0.11% of the labor endowment, accompanied by small reductions in production, search, and private entry labor.

OA.C.5 Restricted Planner: Formulation and Proofs

This appendix formalizes the restricted planner introduced in Section 4.8. The planner maximizes household welfare subject to physical feasibility and the CE's valuation restrictions on product switching, startup operation, and entry. These restrictions preserve the assigned endogenous-exit values that make mature firms' product-line values additive. Assigned exit values create no resources; welfare depends on household consumption.[2]

[2]For a related benchmark, Acemoglu et al. (2018) restrict the planner to decentralized production and pricing while allowing control over entry, exit, and R&D.

OA.C.5.1 Objective and maintained restrictions

Definition OA.1 (Restricted planner)**.** *Given the initial firm distribution, inherited capabilities, and predetermined quantities, the restricted planner chooses pre-entry investments, production, product switching, startup operation, and entry to maximize*

$$W^{\mathrm{SP}} \equiv \max \sum_{t=0}^{\infty} \beta^t \log \overline{C}_t^{\mathrm{SP}}, \tag{OA.C.2}$$

over physically feasible allocations satisfying conditional production efficiency and Assumption OA.1 in the environment below. The fixed-capability problem holds capabilities and associated acquisition costs fixed. The stationary problem requires constant aggregate quantities and distributions.

Planner objects carry the superscript SP. Let ξ collect product-line information, tenure, and common firm characteristics. Operating returns and discounting are defined in (OA.C.14) and (OA.C.16). The product-line and startup value functions below support decisions under the assigned exit-value convention; they are distinct from the planner's welfare value W^{SP}.

Assumption OA.1 (Valuation restrictions)**.** *Conditional on the aggregate path associated with the planner's allocation, product switching decisions, startup operation, and attempted entry maximize net present values in labor units, subject to:*

(i) At ages two and above, each line is evaluated independently of its effects on firm exit and other lines. Upon endogenous exit, each potential line is assigned its vacant-line value $V_t^{ds,\mathrm{SP}}(\emptyset)$. At age one, only the initial line receives this value; initially vacant lines receive zero.

(ii) Startup operation and free entry use these value functions, including assigned exit values. Each entry attempt incurs the resource cost $f_t^{e,\mathrm{SP}}$.

Assigned exit values enter these value functions only. Welfare and feasibility depend on actual consumption and resource use.

The aggregate path is endogenous and must be consistent with the resulting decisions and resource constraints.

Maintained environment. The CE primitives, information structure, aggregation, and timing apply, including $0 < \beta < 1$, $1 < \nu < \eta$, and symmetric product markets. Policies use firms' information

and a common measurable rule for indifference. For each admissible allocation, idiosyncratic uncertainty satisfies an exact law of large numbers conditional on production-planning information, with finite aggregation integrals. Quantities remain measurable with respect to firms' information. The precision path ψ follows the CE law, independently of other lines' retention decisions and aggregate entry. Exogenous death occurs after production with zero scrap value. After updating beliefs, firms dropping all active products exit before search and incur no further search costs; eligible vacant lines of continuing firms search, with positive vacant-line value as in the CE. At age one, retention preserves the other $n(\varphi) - 1$ lines' search opportunities. The initial firm distribution and inherited capabilities are given; incumbents retain their birth-cohort characteristics. Initial quantities y_{ip0} are predetermined by firms' decisions at $t = -1$.

OA.C.5.2 Resources and production

Pre-entry investment. Pre-entry investments determine managerial and productivity capabilities through

$$\log m_{0t}^{\mathrm{SP}} = (1 - \zeta_m) \log \iota_t^{m,\mathrm{SP}} + \zeta_m \log M_{t-1}^{\mathrm{SP}}, \tag{OA.C.3}$$

$$\log z_{0t}^{\mathrm{SP}} = (1 - \zeta_z) \log \iota_t^{z,\mathrm{SP}} + \zeta_z \log Z_{t-1}^{\mathrm{SP}}, \tag{OA.C.4}$$

$$M_{t-1}^{\mathrm{SP}} = m_{0,t-1}^{\mathrm{SP}}, \qquad Z_{t-1}^{\mathrm{SP}} = z_{0,t-1}^{\mathrm{SP}}, \tag{OA.C.5}$$

$$f_t^{e,\mathrm{SP}} = \bar{f}^e + \alpha_m^e (\iota_t^{m,\mathrm{SP}})^\chi + \alpha_z^e (\iota_t^{z,\mathrm{SP}})^\chi. \tag{OA.C.6}$$

At stationarity, $\iota^{m,\mathrm{SP}} = m_0^{\mathrm{SP}}$ and $\iota^{z,\mathrm{SP}} = z_0^{\mathrm{SP}}$. Fixed-capability comparisons hold acquisition costs fixed and use consistent inherited capabilities. The planner chooses pre-entry investments without imposing their private optimality conditions.

Entry and firm distributions. Let $\mathcal{E}_t^{\mathrm{SP}}$ denote attempted entry and $q_t^{e,\mathrm{SP}}$ the probability of operating under (OA.C.23). Entry attempts at t pay $f_t^{e,\mathrm{SP}}$ and, conditional on operation, first produce at $t + 1$. Operating entry and entry resources satisfy

$$\mathcal{M}_t^{e,\mathrm{SP}} = q_t^{e,\mathrm{SP}} \mathcal{E}_t^{\mathrm{SP}}, \qquad \mathcal{F}_t^{e,\mathrm{SP}} = f_t^{e,\mathrm{SP}} \mathcal{E}_t^{\mathrm{SP}}. \tag{OA.C.7}$$

With a single operating cutoff and continuous H_φ, $q_t^{e,\mathrm{SP}} = 1 - H_\varphi(\underline{\varphi}_t^{e,\mathrm{SP}})$; otherwise it is the probability of the operating set under the common indifference rule. Let $\Gamma_t^{e,\mathrm{SP}}$ be the entrant distribution

conditional on operation. Physical firm measures satisfy

$$\Gamma_t^{\mathrm{SP}}(A) = \int_{\mathcal{X}^f} \mathcal{Q}_t^{f,\mathrm{SP}}(A \mid \xi^f)\Gamma_{t-1}^{\mathrm{SP}}(d\xi^f) + \mathcal{M}_t^{e,\mathrm{SP}}\Gamma_t^{e,\mathrm{SP}}(A), \tag{OA.C.8}$$

where Γ_t^{SP} records portfolios after period-t updating, exit, search, and entry planning; its active-line projection $\mu_{p,t}^{\mathrm{SP}}$ determines which lines produce at $t+1$. The kernel $\mathcal{Q}_t^{f,\mathrm{SP}}$ uses the CE's physical law of motion, including both exit mechanisms. The entrant term is zero when $q_t^{e,\mathrm{SP}} = 0$. Accounting retains portfolios and predetermined cohort characteristics without enlarging admissible policy arguments.

Labor feasibility. Operating and search resources equal their unit costs times the mass of producing lines and expected actual searches:

$$\mathcal{F}_t^{o,\mathrm{SP}} = f^o \sum_{p=1}^{N} \mu_{p,t-1}^{\mathrm{SP}}(\mathcal{X}), \qquad \mathcal{F}_t^{s,\mathrm{SP}} = f^s \int \mathcal{N}_t^{s,\mathrm{SP}}(\xi^f)\Gamma_{t-1}^{\mathrm{SP}}(d\xi^f), \tag{OA.C.9}$$

where $\mathcal{N}_t^{s,\mathrm{SP}}(\xi^f)$ counts expected period-t searches per inherited firm after exogenous death and endogenous exit. All three resource costs use period-t labor; the inherited measures index the firms facing current production and subsequent search decisions. Assigned exit values do not enter these resource equations.

Define variable production labor and the productive index of a representative product market by

$$R_t^{\mathrm{SP}} \equiv \overline{L} - \mathcal{F}_t^{o,\mathrm{SP}} - \mathcal{F}_t^{s,\mathrm{SP}} - \mathcal{F}_t^{e,\mathrm{SP}}, \tag{OA.C.10}$$

$$S_t^{\mathrm{SP}} \equiv \int_{\mathcal{X}} \mathcal{B}(\xi)^\eta z_i^{\eta-1} \mu_{p,t-1}^{\mathrm{SP}}(d\xi). \tag{OA.C.11}$$

Here $\mathcal{B}(\xi) \equiv \mathbb{E}[\theta_{ipt}^{(\eta-1)/\eta} \mid \xi]$ is the predictive demand term for production at t, including demand shocks; ξ contains the information available when quantities are chosen. Productivity z_i uses the firm's birth-cohort capability, with $z_i = z_0\varphi_i$ at fixed capabilities. Require $R_t^{\mathrm{SP}} > 0$ and $0 < S_t^{\mathrm{SP}} < \infty$.

Proposition OA.6 (Conditional production and operating returns)**.** *At dates when quantities can be chosen, given active-line measures and variable production labor, the consumption-maximizing allocation satisfies*

$$\ell_{pt}^{\mathrm{SP}}(\xi) = \frac{R_t^{\mathrm{SP}}}{N} \frac{\mathcal{B}(\xi)^\eta z_i^{\eta-1}}{S_t^{\mathrm{SP}}}, \tag{OA.C.12}$$

$$C_{pt}^{\mathrm{SP}} = \frac{R_t^{\mathrm{SP}}}{N}\left(S_t^{\mathrm{SP}}\right)^{1/(\eta-1)}, \qquad \overline{C}_t^{\mathrm{SP}} = N^{\nu/(\nu-1)} C_{pt}^{\mathrm{SP}}. \tag{OA.C.13}$$

The direct marginal contribution of an active line, net of its operating fixed cost and expressed in labor units, is

$$\overline{\pi}_t^{\mathrm{SP}}(\xi) \equiv \frac{R_t^{\mathrm{SP}}}{N(\eta-1)S_t^{\mathrm{SP}}}\mathcal{B}(\xi)^{\eta} z_i^{\eta-1} - f^o = \frac{\ell_{pt}^{\mathrm{SP}}(\xi)}{\eta-1} - f^o. \tag{OA.C.14}$$

At a CE allocation, $\overline{\pi}_t^{\mathrm{SP}}(\xi) = \overline{\pi}_t(\xi)/w_t$.

Proof. Suppress SP. The conditional law of large numbers gives

$$C_{pt}^{(\eta-1)/\eta} = \int \mathcal{B}(\xi)[z_i \ell_{pt}(\xi)]^{(\eta-1)/\eta} \mu_{p,t-1}(d\xi). \tag{OA.C.15}$$

For variable labor L_{pt}, Hölder's inequality bounds the right side by $L_{pt}^{(\eta-1)/\eta} S_{pt}^{1/\eta}$, attained at $\ell_{pt}(\xi) = L_{pt}\mathcal{B}(\xi)^{\eta} z_i^{\eta-1}/S_{pt}$. Symmetry and concavity of the upper CES aggregator give $L_{pt} = R_t/N$, proving (OA.C.12)–(OA.C.13).

Holding search and entry resources fixed, a marginal type-ξ line contributes $\mathcal{B}(\xi)^{\eta} z_i^{\eta-1}/[N(\eta-1)S_t] - f^o/R_t$ to log consumption. Dividing by the marginal value of labor, $1/R_t$, gives (OA.C.14). The CE first-order condition gives expected revenue $\eta w_t \ell_{pt}(\xi)/(\eta-1)$; subtracting labor costs yields the same operating return. This CES alignment is related to Dhingra and Morrow (2019). □

OA.C.5.3 Decision rules under the valuation restrictions

Valuation in labor units. Along a deterministic aggregate path, the value functions use direct marginal-consumption weights $1/R_t^{\mathrm{SP}}$ to discount labor returns:

$$\beta_t^{\mathrm{SP}} \equiv \beta \frac{R_t^{\mathrm{SP}}}{R_{t+1}^{\mathrm{SP}}}. \tag{OA.C.16}$$

This normalization equals β at stationarity. It represents direct marginal-consumption valuation, rather than the costate of the full constrained problem. Appendix OA.C.5.5 relates it to household intertemporal valuation.

Product switching decisions. Under Assumption OA.1, the choice-specific value functions and, at ages two and above, the Bellman equation are:

$$V_t^{ns,\mathrm{SP}}(\xi) = \beta_t^{\mathrm{SP}}\left[\overline{\pi}_{t+1}^{\mathrm{SP}}(\xi) + (1-\delta)\mathbb{E}\left[V_{t+1}^{\mathrm{SP}}(\xi') \mid \xi\right]\right], \tag{OA.C.17}$$

$$V_t^{cs,\mathrm{SP}}(\xi) = -f^s + (1-x)V_t^{ns,\mathrm{SP}}(\xi) + xV_t^{ns,\mathrm{SP}}(\xi^0), \tag{OA.C.18}$$

$$V_t^{ds,\mathrm{SP}}(\emptyset) = -f^s + xV_t^{ns,\mathrm{SP}}(\xi^0) + (1-x)\beta_t^{\mathrm{SP}}(1-\delta)V_{t+1}^{ds,\mathrm{SP}}(\emptyset), \tag{OA.C.19}$$

$$V_t^{\mathrm{SP}}(\xi) = \max\left\{V_t^{ns,\mathrm{SP}}(\xi), V_t^{cs,\mathrm{SP}}(\xi), V_t^{ds,\mathrm{SP}}(\emptyset)\right\}. \tag{OA.C.20}$$

The fresh-product state ξ^0 has the common prior and the same firm characteristics and age. Common states advance in every continuation term, including vacancy, using the CE information transitions. Search costs are incurred after exogenous death and carry no additional $1-\delta$ factor.

Dropping is valued as vacancy continuation whether the firm continues or exits; actual search requires continuation. At ages two and above, the total assigned exit value is $n(\varphi)V_t^{ds,\mathrm{SP}}(\emptyset)$.

Startup operation. At age one, retention preserves the other $n(\varphi)-1$ vacancy opportunities, while exit assigns only the initial line its vacant-line value:

$$V_t^{e,1,\mathrm{SP}}(\xi) = \max\Big\{V_t^{ns,\mathrm{SP}}(\xi) + [n(\varphi)-1]V_t^{ds,\mathrm{SP}}(\emptyset), \\ V_t^{cs,\mathrm{SP}}(\xi) + [n(\varphi)-1]V_t^{ds,\mathrm{SP}}(\emptyset),\ V_t^{ds,\mathrm{SP}}(\emptyset)\Big\}. \tag{OA.C.21}$$

Common states are suppressed; the choice-specific values use age one with continuation at age two.

Before initial production, φ is observed and m is not. Startup operating value and the option to abandon before production satisfy

$$\widetilde{V}_t^{e,\mathrm{SP}}(\varphi) = \beta_t^{\mathrm{SP}}\left[\overline{\pi}_{t+1}^{\mathrm{SP}}(\xi^0(0;\varphi)) + (1-\delta)\mathbb{E}_{m|\varphi}\mathbb{E}^0 V_{t+1}^{e,1,\mathrm{SP}}(\xi')\right], \tag{OA.C.22}$$

$$V_t^{e,\mathrm{SP}}(\varphi) = \max\{\widetilde{V}_t^{e,\mathrm{SP}}(\varphi), 0\}. \tag{OA.C.23}$$

Managerial ability is revealed after initial production; $\mathbb{E}^0$ integrates the subsequent first update. Zero denotes abandonment before production. When the operating decision satisfies single crossing, define $\underline{\varphi}_t^{e,\mathrm{SP}} = \inf\{\varphi : \widetilde{V}_t^{e,\mathrm{SP}}(\varphi) \geq 0\}$; zero-value matching holds only at a continuous interior crossing.

Entry. Conditional on the aggregate path and capability levels, Assumption OA.1(ii) requires

$$\mathcal{E}_t^{\mathrm{SP}} \in \arg\max_{\mathcal{E}\geq 0} \mathcal{E}\left[\mathbb{E}_\varphi V_t^{e,\mathrm{SP}}(\varphi) - f_t^{e,\mathrm{SP}}\right]. \tag{OA.C.24}$$

Linearity in attempted entry implies the complementary slackness condition

$$0 \leq \mathcal{E}_t^{\mathrm{SP}} \perp f_t^{e,\mathrm{SP}} - \mathbb{E}_\varphi V_t^{e,\mathrm{SP}}(\varphi) \geq 0. \tag{OA.C.25}$$

A positive net value makes attempted entry unbounded; a negative value implies zero entry. General equilibrium consistency and feasibility determine entry mass. This condition follows from imposing the valuation criterion on entry, independently of line separability; exit values create no consumption.

OA.C.5.4 Regularity and verification

Assume Borel state spaces, measurable returns and transition kernels, and finite-horizon expected absolute returns uniformly bounded over admissible policies from each initial state and date. Let r_s^{SP} denote the expected next-period operating return discounted by β_s^{SP}, net of current search costs, as in (OA.C.17)–(OA.C.19). The process used to evaluate product-line values includes vacancy continuation after endogenous exit and an absorbing exogenous-death state with zero subsequent returns. For every initial state ξ and date t, assume

$$\lim_{T\to\infty} \sup_g \mathbb{E}_{t,\xi}^g \left[\sum_{s=T}^{\infty} \beta^{s-t} \frac{R_t^{\mathrm{SP}}}{R_s^{\mathrm{SP}}} |r_s^{\mathrm{SP}}| \right] = 0. \tag{OA.C.26}$$

Here g ranges over admissible policies for this process. Ex ante startup values, including vacancy opportunities, are integrable, and admissible aggregate paths satisfy

$$\sum_{t\geq 0} \beta^t |\log \overline{C}_t^{\mathrm{SP}}| < \infty.$$

We assume that the maximum is attained for each planner problem considered.

Lemma OA.5 (Verification of value functions)**.** *For a fixed aggregate path satisfying the stated conditions, the product-line and startup-operation Bellman equations have a finite measurable value function attained by a measurable policy. It is unique among solutions whose discounted expected terminal values vanish under every admissible policy.*

Proof. Backward induction with zero terminal values gives measurable finite-horizon values (Stokey

and Lucas Jr, 1989, Sections 9.1 and 9.4). The discount factors satisfy

$$\prod_{s=t}^{T-1} \beta_s^{\mathrm{SP}} = \beta^{T-t} \frac{R_t^{\mathrm{SP}}}{R_T^{\mathrm{SP}}}. \tag{OA.C.27}$$

Condition (OA.C.26) bounds truncation errors uniformly over policies, yielding convergence to a finite measurable value function and passage through the continuation expectations. Finite action sets and common tie breaking give a measurable maximizer. For the constructed value function, the discounted expected absolute terminal value is bounded by the maximal expected discounted absolute continuation payoff, which vanishes by (OA.C.26). Iterating the Bellman inequality bounds every policy's return, with equality under a maximizing policy. Applying this argument to any solution satisfying the terminal condition proves uniqueness. Integrability determines startup values. □

OA.C.5.5 Replication and capability choice

The following comparison fixes capabilities and acquisition costs; the CE retains startup operation and free entry without imposing private capability-investment optimality.

Proposition OA.7 (Replication and restricted efficiency)**.** *Under the maintained restrictions and regularity conditions:*

(i) For fixed (m_0, z_0) and associated costs, the planner's admissible stationary allocations coincide with stationary CE allocations.

(ii) For a fixed capability path and associated costs, this equivalence extends to paths from a common inherited state if predetermined initial quantities satisfy conditional production, their supporting CE prices satisfy $\overline{P}_0\overline{C}_0 = \eta w_0 R_0/(\eta - 1)$, and decentralized values use household intertemporal valuation.

Within each comparison class, the planner selects a welfare-maximizing CE allocation. Under uniqueness or a common equilibrium-selection restriction,

$$\overline{C}_t^{\mathrm{SP}} = \overline{C}_t^{\mathrm{CE}} \quad \textit{for all } t \geq 0, \qquad W^{\mathrm{SP}} = W^{\mathrm{CE}}. \tag{OA.C.28}$$

Proof. Stationary allocations. Proposition OA.6 gives the CE production allocation and $\overline{\pi}^{\mathrm{SP}} = \overline{\pi}/w$. Dividing the CE decision and entry conditions by the constant wage gives (OA.C.17)–(OA.C.25).

Lemma OA.5 implies

$$V^{j,\mathrm{SP}} = V^j/w, \qquad j \in \{ns, cs, ds, e\}, \qquad V^{\mathrm{SP}} = V/w. \tag{OA.C.29}$$

The normalization also applies to $V^{e,1}$ and $\widetilde{V}^e$. Common tie breaking, physical transitions, and resource constraints make every stationary CE allocation admissible for the planner.

Conversely, any admissible stationary planner allocation is supported by a wage $w > 0$ and CES prices

$$p_{ipt} = \frac{\eta}{\eta - 1} \frac{w}{z_i} \frac{\theta_{ipt}^{(\eta-1)/\eta}}{\mathcal{B}(\xi)}, \qquad P_p = \frac{\eta}{\eta - 1} w \big(S^{\mathrm{SP}}\big)^{-1/(\eta-1)}. \tag{OA.C.30}$$

These prices support the quantities under CES demand and the production first-order condition. Multiplying planner values by w recovers the CE decision and entry conditions. Physical feasibility gives goods and labor clearing and, after consolidating household-owned firms, the household budget identity.

Sequential allocations. Conditional production and the CE quantity first-order condition imply

$$\overline{P}_t \overline{C}_t = N P_{pt} C_{pt} = \frac{\eta}{\eta - 1} w_t R_t. \tag{OA.C.31}$$

The initial supporting condition extends this identity to date zero. The household discount factor in labor units is therefore

$$\beta \frac{\overline{P}_t \overline{C}_t}{\overline{P}_{t+1} \overline{C}_{t+1}} \frac{w_{t+1}}{w_t} = \beta \frac{R_t}{R_{t+1}}. \tag{OA.C.32}$$

Without the initial supporting condition, the first household discount factor in labor units need not equal $\beta R_0/R_1$. The preceding normalization and supporting-price construction apply date by date. Verification gives identical values and policies; the common initial state and physical transitions give identical allocation paths.

Restricted efficiency. Equality of the allocation sets lets the planner select a welfare-maximizing CE. Under uniqueness or the same selection restriction, consumption agrees at every date and welfare convergence gives (OA.C.28). □

Corollary OA.6 (The value of capability choice)**.** *Starting from a stationary CE's firm distribution, inherited capabilities* $(M_{-1}, Z_{-1}) = (m_0^{\mathrm{CE}}, z_0^{\mathrm{CE}})$*, and predetermined quantities, the restricted planner with capability choice attains* $W^{\mathrm{SP}} \geq W^{\mathrm{CE}}$.

Proof. Setting $(\iota_t^{m,\mathrm{SP}}, \iota_t^{z,\mathrm{SP}}) = (m_0^{\mathrm{CE}}, z_0^{\mathrm{CE}})$ at every date reproduces baseline capabilities and costs

through (OA.C.3)–(OA.C.5) and (OA.C.6). Proposition OA.7(i) makes the stationary CE path feasible for the planner, implying $W^{\mathrm{SP}} \geq W^{\mathrm{CE}}$. □

OA.D Empirical Appendix

OA.D.1 Data Construction

Data Sources Our main data source is the U.S. Census Bureau's quinquennial Census of Manufactures (CMF), linked to the Longitudinal Business Database (LBD). Our sample covers eight census years from 1977 to 2012. The CMF covers U.S. manufacturing establishments with at least one paid employee and reports establishment-level employment, payroll, worker hours, input costs, value added, capital expenditures, and inventories. Crucially for our analysis, it reports product codes—five-digit SIC or seven-digit NAICS codes—and the value of shipments by product.[3] See Bernard et al. (2010), Kehrig et al. (2011), and Kehrig and Vincent (2021) for further details.

The LBD provides annual information on U.S. private nonfarm establishments with paid employees from 1976 onward, including employment, payroll, industry codes, and location. We link CMF establishments to the LBD and use firm identifiers to aggregate establishment-level observations to the firm level. The linked data allow us to measure firm size, age, productivity, employment growth, and entry and exit. See Jarmin and Miranda (2002) and Chow et al. (2021) for details.

We use industry-level variables from the NBER-CES Manufacturing Industry Database described in Becker et al. (2013). The version we use covers 1958–2018 and provides annual measures of shipments, value added, employment, payroll, input costs, investment, total factor productivity, and price deflators. Industries are defined by four-digit SIC or six-digit NAICS codes.

In addition, we use retrospective responses for 2005 from the 2010 Management and Organizational Practices Survey (MOPS), an establishment-level survey of U.S. manufacturing.[4] We construct the structured management score, data availability (Q27), and data use intensity (Q28), aggregating establishment-level measures to firms using value-of-shipments weights. We link these measures to the 2007 firm-product observations, whose dropping outcomes cover 2007–2012.

[3] For example, SIC industry 3651 includes radios and radio combinations (36511), television receivers (36512), and consumer high-fidelity components (36514).

[4] See Bloom et al. (2019) for details on the survey and the structured management score.

Product Code Concordances We define products at the five-digit SIC level, using 1987 SIC codes throughout the sample.[5] For 1977–1997, we use product codes that the Census Bureau concords to 1987 SIC codes. To extend this consistent product definition to subsequent census years, we use the 1997 CMF, which reports 1987 SIC and 1997 NAICS product codes.

Starting with the concordance between 1987 SIC and 1997 NAICS product codes, we use the 2002 Census bridge file to link 1997 and 2002 NAICS codes. We repeat this procedure for 2007 and 2012, mapping product codes in each year to 1987 SIC codes. To maintain consistency within firms over 2002–2012, we match firm-product pairs to the preceding census and retain their SIC codes. For pairs without a match, we assign SIC codes using the concordance. When a NAICS product code maps to multiple SIC codes, we assign the SIC code most frequently matched to that NAICS code.

Firm Age and Longitudinal Identifiers We use firm age constructed by the Census Bureau following Haltiwanger et al. (2013). When a firm first appears, its age equals that of its oldest establishment; firm age increases annually thereafter. We define young firms as those aged five or less. Because the LBD begins in 1976, firm ages are left-censored for firms operating that year. We use the 1972 CMF to adjust these ages and reduce misclassification of young firms.[6] We measure firm size by total employment or total value of shipments. All nominal values are deflated to 1997 dollars using industry-level deflators for the relevant variables.

Our baseline analysis uses the Census Bureau's firm identifier, `lbdfid`, which we refer to as the "firm identifier" throughout. As described in Chow et al. (2021), this identifier addresses the recycling of firm identifiers in the legacy LBD but does not fully resolve breaks associated with firm reorganizations. Such breaks can affect our ability to track firms' product portfolios over time. To assess the robustness of our results, we construct alternative longitudinal firm identifiers following Dent et al. (2018).

OA.D.2 Summary Statistics

Table OA.9 summarizes the firm characteristics, product-specific expertise, and management measures used in the empirical analysis. Panel A distinguishes young firms, aged five or less, from

[5] We obtain product-level information from the CMF product trailers, using five-digit SIC codes through 1997 and seven-digit NAICS codes thereafter. For 1977–1997, the files contain both "as-collected" product codes and codes concorded to the 1987 SIC revision.

[6] For LBD firms owning establishments observed in the 1972 CMF, we add four years to firm age if beginning-of-year finished goods inventories (FIB) equal zero and five years if they are positive.

mature firms. Panels B and C report the distributions of expertise and management measures in their respective samples.

Table OA.9: Summary Statistics

	All firms	Young firms	Mature firms
Panel A: Firm characteristics			
Employment	118.8	32.5	131.3
	(1,469.0)	(77.8)	(1,571.0)
Firm age	14.74	2.91	16.44
	(8.42)	(1.62)	(7.59)
Number of products	2.189	1.877	2.233
	(2.896)	(1.509)	(3.041)
Observations	741,000	93,000	648,000
	Mean	Std. dev.	Observations
Panel B: Product-specific expertise			
Normalized expertise, q_{ip}	0.366	0.244	1,622,000
Panel C: Management measures			
Structured management score	0.589	0.159	46,000
Data availability	0.452	0.285	46,000
Intensity of data use	0.432	0.250	46,000

Notes: Panel A reports means and standard deviations (in parentheses) across firm-year observations, including entrants and exiters. The analysis sample consists of firms with estimable product-specific expertise and observations in at least two census years. Young firms are aged five or less; mature firms are aged six or more. Products are defined by five-digit 1987 SIC codes. Panel B reports statistics for normalized expertise across firm-product-year observations in the product-dropping regression sample. Panel C reports firm-level management measures for 2005 from retrospective responses in the 2010 MOPS, aggregated using value-of-shipments weights. All three management measures range from zero to one.

OA.D.3 Additional Descriptive Evidence

Table OA.10 reports the shares of firms engaging in product switching and their output shares by age group, both including and excluding firm entry and exit. To distinguish product switching within continuing firms from changes associated with entry and exit, we focus on the last three columns. In the pooled sample, 65% of continuing firms switch products over a five-year interval, accounting for 91% of their total output. These patterns underscore the prevalence and economic importance of product switching, consistent with Bernard et al. (2010).

Product switching often involves both adding and dropping products, which we refer to as product churning. Approximately 52% of product-switching firms (34% out of 65%) both add and drop products; the remainder either add or drop products. Product churning is more prevalent among

Table OA.10: Share of Product Switching Firms

variables	Including entry and exit			Continuers only		
	All	Age 0-5	Age 6 +	All	Age 0-5	Age 6 +
			Product adding only			
Share of firms	0.094	0.100	0.093	0.128	0.130	0.128
Output share	0.058	0.098	0.057	0.067	0.139	0.066
			Product dropping only			
Share of firms	0.398	0.329	0.407	0.182	0.143	0.187
Output share	0.282	0.404	0.280	0.169	0.146	0.169
			Product churning			
Share of firms	0.248	0.310	0.239	0.336	0.394	0.327
Output share	0.582	0.284	0.588	0.674	0.409	0.678

Notes: Firms in the manufacturing sector from 1982 to 2007.

Table OA.11: Share of Multi-Product Firms

Classification	Share of firms			Output share		
	All	Age 0-5	Age 6 +	All	Age 0-5	Age 6 +
sic5	0.450	0.432	0.452	0.891	0.615	0.896
sic4	0.332	0.301	0.336	0.838	0.451	0.845
sic2	0.183	0.153	0.187	0.702	0.267	0.711

Notes: Firms in the manufacturing sector from 1982 to 2007.

young firms, whereas dropping products without adding others is less common than among mature firms.

Table OA.11 reports the prevalence of multi-product firms at three aggregation levels: five-digit SIC products, four-digit SIC industries, and two-digit SIC sectors. Under our baseline five-digit definition, multi-product firms constitute 45% of firms but account for 89% of output. Multi-product firms also span multiple industries and sectors. They are prevalent even among young firms, accounting for 43% of firms and 62% of output in this age group. Single-product firms, by contrast, constitute 55% of firms but account for 11% of output. Firms that produce a single product and never switch products during the sample represent 14.8% of firms and 2.8% of output.

Table OA.12 reports the means and standard deviations of the number of products firms produce, add, and drop, for continuing firms, including single-product firms. On average, firms produce 2.2 products and add 1.2 and drop 0.9 products over five years.[7] Young firms produce fewer products than mature firms (1.6 versus 2.3), but add more (1.6 versus 1.2) and drop fewer (0.8 versus 1.0).

[7] Bernard et al. (2010) report an average of 3.5 products among multi-product firms.

Table OA.12: Number of Products Produced/Added/Dropped

	Including entry and exit			Continuers only		
	All	Age 0-5	Age 6 +	All	Age 0-5	Age 6 +
			Number of products firms produce			
Mean	2.189	1.877	2.233	2.192	1.645	2.269
Stdev.	(2.896)	(1.509)	(3.041)	(3.153)	(1.377)	(3.321)
			Number of products firms add			
Mean	1.27	1.831	1.189	1.244	1.605	1.194
Stdev.	(1.831)	(1.518)	(1.858)	(1.880)	(1.374)	(1.935)
			Number of products firms drop			
Mean	1.273	1.218	1.281	0.941	0.790	0.963
Stdev.	(1.981)	(1.482)	(2.043)	(1.841)	(1.101)	(1.922)

Notes: Firms in the manufacturing sector from 1982 to 2007.

Table OA.13: Product Add and Retention

	$Add_{t:t+5}$	$Add^{high}_{t:t+5}$
Young Firm$_{ijt}$	0.148***	-0.112***
	(0.005)	(0.017)
Observations	682,000	66,000
Fixed effects	i, jt	i, jt
Controls	Full	Full

Notes: Estimates for industry-year, firm fixed effects, and the constant are suppressed. Robust standard errors clustered at the firm's modal five-digit SIC product level are reported in parentheses. Observations are unweighted, and observation counts are rounded due to the Census Bureau disclosure avoidance procedures. * p<0.1, ** p<0.05, *** p<0.01.

OA.D.4 Product Additions over the Firm Life cycle

We examine how the probability of adding products and the expertise of these additions vary with firm age. Let $\mathcal{I}^{add}_{ijt}$ equal one if firm i adds at least one product between census years t and $t+5$, following the definition in Section 3.1. To assess the expertise of these additions, we define $\mathcal{I}^{add,high}_{ijt}$ to equal one if the average expertise of newly added products observed at $t+5$ exceeds the average expertise of the firm's products at t. For each outcome, we estimate:

$$Y_{ijt} = \alpha + \beta \mathcal{I}^{young}_{ijt} + X_{ijt}\gamma + \delta_i + \delta_{jt} + \varepsilon_{ijt}, \tag{OA.D.1}$$

where $Y_{ijt} \in \left\{\mathcal{I}^{add}_{ijt}, \mathcal{I}^{add,high}_{ijt}\right\}$, and δ_{jt} denotes an industry-year fixed effect.

Column 1 of Table OA.13 shows that young firms are 14.8 percentage points more likely to add a product over the next five years. Column 2 shows that, in the sample used to assess addition quality,

Table OA.14: Product Add/Drop and Firm Performance

	$\Delta Vadd$	ΔEmp	ΔTFP	ΔLP
$Add_{t:t+5}$	0.830***	0.802***	0.735***	0.762***
	(0.019)	(0.019)	(0.019)	(0.019)
$Drop_{t:t+5}$	-0.754***	-0.731***	-0.686***	-0.713***
	(0.019)	(0.018)	(0.017)	(0.018)
Observations	682,000	682,000	648,000	648,000
Fixed effects	i, jt	i, jt	i, jt	i, jt
Controls	Full	Full	Full	Full

Notes: Estimates for industry-year, firm fixed effects, and the constant are suppressed. Robust standard errors clustered at the firm's modal five-digit SIC product level are reported in parentheses. Observations are unweighted, and observation counts are rounded due to the Census Bureau disclosure avoidance procedures. * p<0.1, ** p<0.05, *** p<0.01.

young firms are 11.2 percentage points less likely to have additions whose average expertise exceeds that of their existing portfolio.[8]

Together, these results suggest that young firms experiment with products but exhibit weaker selection on expertise. These patterns are consistent with the model's learning mechanism: lower information-processing ability limits firms' ability to distinguish high-expertise from low-expertise products early in the firm life cycle.

OA.D.5 Product Additions, Drops, and Firm Performance

To examine how product portfolio adjustments relate to firm performance, we estimate:

$$\Delta y_{ijt} = \alpha + \beta_1 \mathcal{I}_{ijt}^{drop} + \beta_2 \mathcal{I}_{ijt}^{add} + X_{ijt}\gamma + \delta_i + \delta_{jt} + \varepsilon_{ijt}, \tag{OA.D.2}$$

where Δy_{ijt} denotes the change in log real value added, employment, total factor productivity, or labor productivity between t and $t+5$.[9] The indicators $\mathcal{I}_{ijt}^{drop}$ and $\mathcal{I}_{ijt}^{add}$ equal one if firm i drops or adds at least one product over the same interval, respectively.

Table OA.14 shows that product additions are positively associated with growth across all four performance measures, while product drops are negatively associated with growth.[10] Under this additive specification, firms that both add and drop products have higher predicted growth than

[8] Full regression results appear in Appendix OA.D.8.

[9] Value added equals Total Value of Shipments + Final Inventory Investment + Work-in-Progress Goods - Resales - Material Inputs - Energy Expenditures. Labor productivity is real value of shipments divided by total employment. We use the Census Bureau's TFP beta 8 measure available in the CMF.

[10] Appendix Table OA.24 reports full regression results.

Table OA.15: Firm-Level Output and Long-run Performance Growth Decomposition

	Output			Long-run Performance (θ_{ip})		
	all	age 0-5	age 6 +	all	age 0-5	age 6 +
Total changes	0.112	0.662	0.104	0.017	0.130	0.016
Within share	0.543	0.444	0.545	0	0	0
Between share	0.008	0.002	0.009	0.318	0.126	0.320
Cross share	0.009	-0.005	0.010	0	0	0
Addition share	0.424	0.554	0.422	0.574	0.853	0.572
Drop share	0.015	0.005	0.015	0.107	0.022	0.108

Notes: Firms in the manufacturing sector from 1982 to 2007.

firms that do neither. The implied differences, $\hat{\beta}_1 + \hat{\beta}_2$, are 0.076 log points for real value added, 0.071 for employment, and 0.049 for both total factor productivity and labor productivity.[11]

OA.D.6 Decomposition of Firm-Level Growth in Output and Product Expertise

We conduct an accounting decomposition of changes in the shipment-weighted average of log real product shipments and product-specific expertise. This decomposition distinguishes changes within continuing firm-product pairs, reallocation across continuing products, and product additions and drops.

Define within-firm output share of product p at t for firm i as $\omega_{ipt} = \frac{PV_{ipt}}{\sum_{p\in\mathcal{P}_{it}} PV_{ipt}}$, where PV_{ipt} denotes the real product value of shipments of product p by firm i at t, and $\mathcal{P}_{it}$ is the set of products in firm i's product portfolio at t. Let y_{ipt} denote either the log of real product value of shipments, $\log(PV_{ipt})$, or the firm-product long-run performance, θ_{ip}. We compute the shipment-weighted average log product shipments or expertise as $y_{it} = \sum_{p\in\mathcal{P}_{it}} \omega_{ipt} y_{ipt}$.

We then decompose the change in each shipment-weighted average as:

$$\begin{aligned}\Delta y_{it} = &\underbrace{\sum_{p\in\mathcal{C}_{it}} \omega_{ip,t-5}\Delta y_{ipt}}_{\text{within term}} + \underbrace{\sum_{p\in\mathcal{C}_{it}} (y_{ip,t-5} - y_{it-5})\Delta\omega_{ipt}}_{\text{between term}} + \underbrace{\sum_{p\in\mathcal{C}_{it}} \Delta\omega_{ipt}\Delta y_{ipt}}_{\text{cross term}} \\ &+ \underbrace{\sum_{p\in\mathcal{N}_{it}} \omega_{ipt}(y_{ipt} - y_{it-5})}_{\text{addition term}} - \underbrace{\sum_{p\in\mathcal{X}_{it}} \omega_{ip,t-5}(y_{ip,t-5} - y_{it-5})}_{\text{drop term}}.\end{aligned} \tag{OA.D.3}$$

$\Delta X_{\cdot t} = X_{\cdot t} - X_{\cdot t-5}$ is the change in any variable $X_{\cdot t}$ between $t-5$ and t, $\mathcal{C}_{it}$ is the set of continuing

[11] For related evidence linking product switching to firm growth and productivity, see Bernard et al. (2010) and Argente et al. (2018).

products that firm i produces in both $t-5$ and t, $\mathcal{N}_{it}$ is the set of products newly added by firm i in t, and $\mathcal{X}_{it}$ is the set of products dropped by firm i in t.

Table OA.15 presents decomposition results for shipment-weighted average log product shipments and expertise (θ_{ip}). We normalize each term by the total change to quantify its contribution to growth. We refer to these values as shares. The first three columns show that growth in the shipment-based measure is primarily driven by within-product growth (within share) and new product addition (addition term). For young firms, however, new product addition plays a more prominent role than within-product growth. The combined contribution of product-switching components, addition and drop shares, is 28% higher for young firms than for mature firms (0.554 + 0.005 vs. 0.422 + 0.015).

OA.D.7 Full Tables: Product Dropping

The main text presents the product-dropping regressions through the interacted specification in equation (3.41). The specifications for overall selection on performance and average age differences are, respectively,

$$\mathcal{I}_{ipjt}^{drop} = \alpha + \beta Q_{ipt} + X_{ipjt}\gamma_1 + X_{ijt}\gamma_2 + \delta_i + \delta_{pt} + \varepsilon_{ipjt}. \tag{OA.D.4}$$

$$\mathcal{I}_{ipjt}^{drop} = \alpha + \beta \mathcal{I}_{ijt}^{young} + X_{ipjt}\gamma_1 + X_{ijt}\gamma_2 + \delta_i + \delta_{pt} + \varepsilon_{ipjt}, \tag{OA.D.5}$$

where the performance measures, controls, and fixed effects are defined in the main text. For management and information processing, we estimate

$$\begin{aligned}\mathcal{I}_{ipjt}^{drop} = \alpha &+ \beta_1 q_{ip} + \beta_2 m_{ijt} + \beta_3 q_{ip} \times m_{ijt} + \lambda_1 D_{ijt} + \lambda_2 q_{ip} \times D_{ijt} \\ &+ X_{ipjt}\gamma_1 + X_{ijt}\gamma_2 + \delta_i + \delta_{pt} + \varepsilon_{ipjt},\end{aligned} \tag{OA.D.6}$$

where m_{ijt} is the 2005 structured management score, data availability (Q27), or data use intensity (Q28), interacted with $\mathbf{1}\{t = 2007\}$. These specifications retain the pooled product-dropping panel. Missing measures are set to zero, and D_{ijt} indicates missing observations for the corresponding measure. Including D_{ijt} and $q_{ip} \times D_{ijt}$ allows both the intercept and the expertise slope to differ for missing observations. The management main effect remains in the regression but is not reported in the tables.

Table OA.16: Probability of Product Drop (PV)

VARIABLES	$Drop_{t:t+5}$	$Drop_{t:t+5}$	$Drop_{t:t+5}$	$Drop_{t:t+5}$	$Drop_{t:t+5}$
$\log(PV_{ipt})$	-0.055***	-0.054***	-0.053***	-0.053***	-0.052***
	(0.002)	(0.002)	(0.002)	(0.002)	(0.002)
log(Emp)			-0.020***	-0.009***	-0.012***
			(0.002)	(0.002)	(0.002)
log(Total Assets)			0.003***	0.019***	0.019***
			(0.001)	(0.004)	(0.004)
# establishments			-0.002***	-0.002***	-0.002***
			(0.000)	(0.000)	(0.000)
# products			0.007***	0.007***	0.006***
			(0.000)	(0.000)	(0.000)
log(CA)			-0.002***	-0.002***	-0.002***
			(0.001)	(0.001)	(0.001)
log(CS)			-0.004***	-0.004***	-0.003***
			(0.001)	(0.001)	(0.001)
TFP			-0.004*	-0.006**	-0.007***
			(0.002)	(0.002)	(0.003)
log(Prod. Emp)				-0.009***	-0.010***
				(0.001)	(0.001)
log(Equipment)				-0.028***	-0.029***
				(0.004)	(0.004)
log(Structure)				0.011***	0.011***
				(0.003)	(0.003)
log(Material)				-0.004***	-0.004***
				(0.001)	(0.001)
log(Energy)				0.004***	0.003***
				(0.001)	(0.001)
Tenure		0.004***			0.004***
		(0.000)			(0.001)
Tenure squared		-0.000***			-0.000***
		(0.000)			(0.000)
Constant	0.943***	0.942***	0.965***	0.978***	1.004***
	(0.010)	(0.010)	(0.015)	(0.016)	(0.017)

Notes: Estimates for product-year, firm fixed effects, and the constant are suppressed. Observations are unweighted. * p<0.1, ** p<0.05, *** p<0.01.

Table OA.17: Probability of Product Drop (θ_{ip})

VARIABLES	$Drop_{t:t+5}$	$Drop_{t:t+5}$	$Drop_{t:t+5}$	$Drop_{t:t+5}$	$Drop_{t:t+5}$
θ_{ip}	-1.212***	-1.028***	-1.189***	-1.187***	-1.074***
	(0.029)	(0.022)	(0.029)	(0.029)	(0.021)
log(Emp)			-0.035***	-0.015***	-0.016***
			(0.002)	(0.002)	(0.002)
log(Total Assets)			0.002*	0.023***	0.021***
			(0.001)	(0.004)	(0.004)
# establishments			-0.002***	-0.002***	-0.002***
			(0.000)	(0.000)	(0.000)
# products			0.005***	0.005***	0.005***
			(0.000)	(0.000)	(0.000)
log(CA)			0	0	0.002***
			(0.001)	(0.001)	(0.001)
log(CS)			-0.005***	-0.005***	-0.002***
			(0.001)	(0.001)	(0.001)
TFP			-0.011***	-0.020***	-0.013***
			(0.002)	(0.002)	(0.002)
log(Prod. Emp)				-0.009***	-0.010***
				(0.001)	(0.001)
log(Equipment)				-0.025***	-0.025***
				(0.004)	(0.004)
log(Structure)				0.005**	0.006**
				(0.003)	(0.003)
log(Material)				-0.013***	-0.008***
				(0.001)	(0.001)
log(Energy)				0.001	0.002*
				(0.001)	(0.001)
$\log(PV_{ipt})$		-0.021***			-0.017***
		(0.001)			(0.001)
Tenure		0.041***			0.042***
		(0.001)			(0.001)
Tenure squared		-0.001***			-0.001***
		(0.000)			(0.000)
Constant	1.053***	0.963***	1.164***	1.221***	1.111***
	(0.015)	(0.014)	(0.021)	(0.022)	(0.022)

Notes: Estimates for product-year, firm fixed effects, and the constant are suppressed. Observations are unweighted. * p<0.1, ** p<0.05, *** p<0.01.

Table OA.18: Probability of Product Drop (Young Firm)

VARIABLES	$Drop_{t:t+5}$	$Drop_{t:t+5}$	$Drop_{t:t+5}$	$Drop_{t:t+5}$	$Drop_{t:t+5}$
Young Firm$_{it}$	-0.084***	-0.073***	-0.092***	-0.093***	-0.075***
	(0.004)	(0.003)	(0.004)	(0.004)	(0.003)
log(Emp)			-0.050***	-0.025***	-0.015***
			(0.002)	(0.002)	(0.002)
log(Total Assets)			-0.008***	0.015***	0.019***
			(0.001)	(0.005)	(0.004)
# establishments			-0.004***	-0.004***	-0.002***
			(0.000)	(0.000)	(0.000)
# products			0.011***	0.011***	0.006***
			(0.001)	(0.001)	(0.000)
log(CA)			-0.003***	-0.002***	-0.001**
			(0.001)	(0.001)	(0.001)
log(CS)			-0.002**	-0.002**	-0.003***
			(0.001)	(0.001)	(0.001)
TFP			-0.014***	-0.027***	-0.008***
			(0.002)	(0.003)	(0.003)
log(Prod. Emp)				-0.008***	-0.010***
				(0.001)	(0.001)
log(Equipment)				-0.032***	-0.029***
				(0.004)	(0.004)
log(Structure)				0.011***	0.011***
				(0.003)	(0.003)
log(Material)				-0.019***	-0.005***
				(0.001)	(0.001)
log(Energy)				-0.002*	0.003***
				(0.001)	(0.001)
$\log(PV_{ipt})$		-0.054***			-0.052***
		(0.002)			(0.002)
Tenure		0.002***			0.003***
		(0.000)			(0.000)
Tenure squared		-0.000***			-0.000***
		(0.000)			(0.000)
Constant	0.591***	0.951***	0.828***	0.918***	1.038***
	(0.000)	(0.011)	(0.015)	(0.016)	(0.018)

Notes: Estimates for product-year, firm fixed effects, and the constant are suppressed. Observations are unweighted. * p<0.1, ** p<0.05, *** p<0.01.

Table OA.19: Probability of Product Drop (PV $\times$ Young)

VARIABLES	$Drop_{t:t+5}$	$Drop_{t:t+5}$	$Drop_{t:t+5}$	$Drop_{t:t+5}$	$Drop_{t:t+5}$
$\log(PV_{ipt})$	-0.056***	-0.055***	-0.054***	-0.054***	-0.052***
	(0.002)	(0.002)	(0.002)	(0.002)	(0.002)
Young Firm$_{it}$	-0.163***	-0.142***	-0.159***	-0.160***	-0.141***
	(0.008)	(0.007)	(0.008)	(0.008)	(0.007)
$\log(PV_{ipt}) \times$ Young$_{it}$	0.012***	0.011***	0.011***	0.011***	0.010***
	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
log(Emp)			-0.025***	-0.013***	-0.015***
			(0.002)	(0.002)	(0.002)
log(Total Assets)			0.001	0.019***	0.019***
			(0.001)	(0.004)	(0.004)
# establishments			-0.002***	-0.002***	-0.002***
			(0.000)	(0.000)	(0.000)
# products			0.007***	0.007***	0.006***
			(0.000)	(0.000)	(0.000)
log(CA)			-0.002***	-0.001*	-0.001*
			(0.001)	(0.001)	(0.001)
log(CS)			-0.003***	-0.003***	-0.003***
			(0.001)	(0.001)	(0.001)
TFP			-0.005**	-0.008***	-0.009***
			(0.002)	(0.003)	(0.003)
log(Prod. Emp)				-0.009***	-0.010***
				(0.001)	(0.001)
log(Equipment)				-0.029***	-0.029***
				(0.004)	(0.004)
log(Structure)				0.011***	0.011***
				(0.003)	(0.003)
log(Material)				-0.004***	-0.005***
				(0.001)	(0.001)
log(Energy)				0.004***	0.003***
				(0.001)	(0.001)
Tenure		0.003***			0.003***
		(0.000)			(0.000)
Tenure squared		-0.000***			-0.000***
		(0.000)			(0.000)
Constant	0.960***	0.957***	1.012***	1.027***	1.044***
	(0.011)	(0.011)	(0.016)	(0.017)	(0.018)

Notes: Estimates for product-year, firm fixed effects, and the constant are suppressed. Observations are unweighted. * p<0.1, ** p<0.05, *** p<0.01.

Table OA.20: Probability of Product Drop ($\theta_{ip} \times$ Young)

VARIABLES	$Drop_{t:t+5}$	$Drop_{t:t+5}$	$Drop_{t:t+5}$	$Drop_{t:t+5}$	$Drop_{t:t+5}$
θ_{ip}	-1.221***	-1.032***	-1.196***	-1.194***	-1.081***
	(0.030)	(0.022)	(0.030)	(0.030)	(0.022)
Young Firm$_{it}$	-0.411***	-0.371***	-0.402***	-0.403***	-0.368***
	(0.022)	(0.023)	(0.022)	(0.022)	(0.022)
$\theta_{ip} \times$ Young$_{it}$	0.395***	0.514***	0.367***	0.368***	0.497***
	(0.044)	(0.043)	(0.043)	(0.043)	(0.043)
log(Emp)			-0.040***	-0.020***	-0.018***
			(0.002)	(0.002)	(0.002)
log(Total Assets)			0	0.022***	0.020***
			(0.001)	(0.004)	(0.004)
# establishments			-0.002***	-0.002***	-0.001***
			(0.000)	(0.000)	(0.000)
# products			0.005***	0.005***	0.005***
			(0.000)	(0.000)	(0.000)
log(CA)			0	0.001*	0.002***
			(0.001)	(0.001)	(0.001)
log(CS)			-0.004***	-0.004***	-0.002***
			(0.001)	(0.001)	(0.001)
TFP			-0.012***	-0.021***	-0.014***
			(0.002)	(0.002)	(0.002)
log(Prod. Emp)				-0.009***	-0.010***
				(0.001)	(0.001)
log(Equipment)				-0.026***	-0.026***
				(0.004)	(0.004)
log(Structure)				0.005*	0.006**
				(0.003)	(0.003)
log(Material)				-0.014***	-0.008***
				(0.001)	(0.001)
log(Energy)				0	0.002*
				(0.001)	(0.001)
$\log(PV_{ipt})$		-0.021***			-0.017***
		(0.001)			(0.001)
Tenure		0.039***			0.039***
		(0.001)			(0.001)
Tenure squared		-0.001***			-0.001***
		(0.000)			(0.000)
Constant	1.071***	0.980***	1.222***	1.280***	1.146***
	(0.015)	(0.014)	(0.021)	(0.023)	(0.022)

Notes: Estimates for product-year, firm fixed effects, and the constant are suppressed. Observations are unweighted. * p<0.1, ** p<0.05, *** p<0.01.

Table OA.21: Probability of Product Drop ($q_{ip} \times m_{ijt}$)

VARIABLES	$Drop_{t:t+5}$	$Drop_{t:t+5}$	$Drop_{t:t+5}$	$Drop_{t:t+5}$	$Drop_{t:t+5}$	$Drop_{t:t+5}$
$q_{ip} \times m_{ijt}$	-0.983***	-1.036***	-0.768***	-0.832***	-0.837***	-0.912***
	(0.080)	(0.082)	(0.061)	(0.062)	(0.066)	(0.068)
q_{ip}	-1.211***	-1.055***	-1.292***	-1.127***	-1.289***	-1.119***
	(0.045)	(0.041)	(0.037)	(0.033)	(0.038)	(0.035)
log(Emp)		-0.015***		-0.015***		-0.015***
		(0.002)		(0.002)		(0.002)
log(Total Assets)		0.021***		0.020***		0.020***
		(0.004)		(0.004)		(0.004)
# establishments		-0.001***		-0.002***		-0.002***
		(0.000)		(0.000)		(0.000)
# products		0.005***		0.005***		0.005***
		(0.000)		(0.000)		(0.000)
log(CA)		0.002***		0.002***		0.002***
		(0.001)		(0.001)		(0.001)
log(CS)		-0.001		-0.001		-0.001
		(0.001)		(0.001)		(0.001)
TFP		-0.012***		-0.012***		-0.012***
		(0.002)		(0.002)		(0.002)
log(Prod. Emp)		-0.011***		-0.011***		-0.011***
		(0.001)		(0.001)		(0.001)
log(Equipment)		-0.025***		-0.025***		-0.025***
		(0.004)		(0.004)		(0.004)
log(Structure)		0.005**		0.005**		0.005**
		(0.003)		(0.003)		(0.003)
log(Material)		-0.007***		-0.007***		-0.007***
		(0.001)		(0.001)		(0.001)
log(Energy)		0.002***		0.002***		0.002***
		(0.001)		(0.001)		(0.001)
$\log(PV_{ipt})$		-0.017***		-0.017***		-0.017***
		(0.001)		(0.001)		(0.001)
Tenure		0.041***		0.041***		0.041***
		(0.001)		(0.001)		(0.001)
Tenure squared		-0.001***		-0.001***		-0.001***
		(0.000)		(0.000)		(0.000)
Constant	1.014***	1.048***	0.998***	1.056***	1.012***	1.064***
	(0.022)	(0.028)	(0.019)	(0.025)	(0.019)	(0.026)

Notes: Columns (1)–(2) use the structured management score of Bloom et al. (2019), columns (3)–(4) use data availability (Q27), and columns (5)–(6) use data use intensity (Q28). Measures refer to 2005, reported in the 2010 MOPS, and enter interacted with the 2007 indicator. Missing measures are set to zero; a missing-measure indicator and its interaction with normalized expertise q_{ip} are included. Management main effects, missingness controls, and firm and product-year fixed effects are not reported. Standard errors clustered at the five-digit SIC product level appear in parentheses. Observations are unweighted. * $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$.

OA.D.8 Full Tables: Firm-Level Evidence

Table OA.22: Product Add and Retention (Young Firm)

	$Add_{t:t+5}$	$Add_{t:t+5}$	$Add_{t:t+5}$
Young Firm$_{it}$	0.140***	0.148***	0.148***
	(0.005)	(0.005)	(0.005)
log(Emp)		0.032***	0.029***
		(0.003)	(0.002)
log(Total Assets)		0.010***	-0.005
		(0.001)	(0.004)
# establishments		0.012***	0.012***
		(0.004)	(0.003)
# products		-0.027***	-0.027***
		(0.002)	(0.002)
log(CA)		-0.003***	-0.003**
		(0.001)	(0.001)
log(CS)		-0.006***	-0.007***
		(0.001)	(0.001)
TFP		-0.001	0.008***
		(0.003)	(0.003)
log(Prod. Emp)			-0.008***
			(0.001)
log(Equipment)			0.021***
			(0.003)
log(Structure)			-0.008***
			(0.003)
log(Material)			0.010***
			(0.001)
log(Energy)			0.006***
			(0.001)
Constant	0.350***	0.220***	0.174***
	(0.002)	(0.015)	(0.018)

Notes: Estimates for industry-year, firm fixed effects, and the constant are suppressed. Observations are unweighted. * p<0.1, ** p<0.05, *** p<0.01.

Table OA.23: High-Performing Product Add and Retention (Young Firm)

	$Add^{high}_{t:t+5}$	$Add^{high}_{t:t+5}$	$Add^{high}_{t:t+5}$
Young Firm$_{it}$	-0.115***	-0.114***	-0.112***
	(0.016)	(0.017)	(0.017)
log(Emp)		-0.018**	-0.018**
		(0.008)	(0.009)
log(Total Assets)		-0.007	0.037***
		(0.005)	(0.012)
# establishments		-0.007***	-0.007***
		(0.003)	(0.003)
# products		0.020***	0.020***
		(0.003)	(0.003)
log(CA)		0.003	0.002
		(0.003)	(0.003)
log(CS)		-0.004	-0.003
		(0.003)	(0.003)
TFP		-0.020**	-0.034***
		(0.008)	(0.009)
log(Prod. Emp)			0.021***
			(0.003)
log(Equipment)			-0.025***
			(0.009)
log(Structure)			-0.015*
			(0.009)
log(Material)			-0.016***
			(0.004)
log(Energy)			-0.013***
			(0.005)
Constant	0.347***	0.463***	0.516***
	(0.003)	(0.041)	(0.049)

Notes: Estimates for industry-year, firm fixed effects, and the constant are suppressed. Robust standard errors clustered at the firm's modal five-digit SIC product level are reported in parentheses. Observations are unweighted, and observation counts are rounded due to the Census Bureau disclosure avoidance procedures. * p<0.1, ** p<0.05, *** p<0.01.

Table OA.24: Product Add/Drop and Firm Performance

	$\Delta Vadd$	ΔEmp	ΔTFP	ΔLP
$Add_{t:t+5}$	0.830***	0.802***	0.735***	0.762***
	(0.019)	(0.019)	(0.019)	(0.019)
$Drop_{t:t+5}$	-0.754***	-0.731***	-0.686***	-0.713***
	(0.019)	(0.018)	(0.017)	(0.018)
log(Emp)	-0.187***	-0.604***	0.099***	0.200***
	(0.009)	(0.008)	(0.007)	(0.007)
log(Total Assets)	-0.071***	-0.021***	-0.026**	-0.018
	(0.006)	(0.006)	(0.012)	(0.011)
# establishments	-0.027***	-0.024***	-0.026***	-0.028***
	(0.008)	(0.007)	(0.008)	(0.008)
# products	0.077***	0.079***	0.070***	0.072***
	(0.006)	(0.005)	(0.005)	(0.006)
log(CA)	-0.003*	0.006***	0.002	0.003
	(0.002)	(0.002)	(0.002)	(0.002)
log(CS)	0.014***	0.014***	0.011***	0.012***
	(0.002)	(0.002)	(0.002)	(0.002)
TFP	-0.561***	0.130***	-0.369***	-0.088***
	(0.012)	(0.006)	(0.022)	(0.026)
log(Prod. Emp)	0.021***	0.028***	0.019***	0.001
	(0.003)	(0.003)	(0.003)	(0.003)
log(Equipment)	0.030***	0.075***	0.059***	0.029***
	(0.005)	(0.005)	(0.007)	(0.006)
log(Structure)	-0.046***	-0.028***	-0.028***	-0.035***
	(0.005)	(0.005)	(0.008)	(0.007)
log(Material)	-0.049***	0.068***	0.023***	-0.042***
	(0.006)	(0.003)	(0.006)	(0.008)
log(Energy)	-0.030***	0.042***	0.010***	0.035***
	(0.004)	(0.002)	(0.003)	(0.004)
Constant	2.252***	0.424***	-0.415***	-0.633***
	(0.052)	(0.047)	(0.081)	(0.100)
Observations	682,000	682,000	648,000	648,000
Fixed effects	i, jt	i, jt	i, jt	i, jt
Controls	Full	Full	Full	Full

Notes: Estimates for industry-year, firm fixed effects, and the constant are suppressed. Robust standard errors clustered at the firm's modal five-digit SIC product level are reported in parentheses. Observations are unweighted, and observation counts are rounded due to the Census Bureau disclosure avoidance procedures. * p<0.1, ** p<0.05, *** p<0.01.

OA.D.9 Robustness

First, differences in observation histories may affect the precision of estimated expertise, particularly for young firms. We therefore re-estimate expertise using only the first two observations per firm-product pair, holding the number of observations constant across pairs. We also use 5-, 10-, and 15-year rolling estimation windows to assess sensitivity to the horizon over which expertise is assumed constant. The interaction between expertise and the young-firm indicator remains positive and statistically significant across these specifications, although its magnitude varies.

Second, expertise estimates may capture firm-level determinants of shipments not fully accounted for by baseline controls. We vary firm-level controls, including a specification without them. We also consider replacing these controls with firm-year fixed effects, which absorb time-varying conditions common to a firm's products. Alternatively, we consider an AKM-style specification (Abowd et al., 1999) and construct product-specific expertise as the time average of residuals within firm-product pairs.[12] The positive age interaction persists when expertise is estimated without firm-level controls.

Third, we assess sensitivity to expertise normalization. We replace the baseline measure with standardized estimates and percentile rankings within industry-year or product-year cells. These alternatives measure expertise relative to firms and products within these cells. The age difference in selection persists across these measures.

Fourth, we assess statistical inference using bootstrapped standard errors. The expertise coefficient remains negative and the interaction with the young-firm indicator positive, with both statistically significant.

Finally, product disappearance may reflect firm exit rather than portfolio adjustment within continuing firms. We re-estimate the product-dropping regressions after excluding firms that subsequently exit. The main age-related selection pattern persists.

[12] Appendix OA.D.11 discusses these specifications.

Table OA.25: Probability of Product Drop (Alternative θ_{ip} Estimates)

VARIABLES	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
θ_{ip}	0.037	-0.013	-0.608***	-0.617***	-0.491***	-0.495***	-0.313***	-0.318***
	(0.037)	(0.037)	(0.016)	(0.016)	(0.015)	(0.015)	(0.016)	(0.016)
Young Firm		-0.453***		-0.344***		-0.303***		-0.130***
		(0.025)		(0.016)		(0.017)		(0.020)
$\theta_{ip} \times$ Young		0.439***		0.376***		0.287***		0.083***
		(0.042)		(0.025)		(0.026)		(0.030)
Constant	0.781***	0.849***	0.946***	0.983***	0.897***	0.935***	0.939***	0.974***
	(0.021)	(0.020)	(0.021)	(0.021)	(0.019)	(0.019)	(0.016)	(0.017)
Observations	1,622,000	1,622,000	1,622,000	1,622,000	1,622,000	1,622,000	1,622,000	1,622,000
Controls	Full	Full	Full	Full	Full	Full	Full	Full
θ_{ip} estimation	2obs	2obs	5yr rolling	5yr rolling	10yr rolling	10yr rolling	15yr rolling	15yr rolling

VARIABLES	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
θ_{ip}	-0.050***	-0.050***	-0.158***	-0.160***	-0.049***	-0.050***	-0.160***	-0.163***
	(0.001)	(0.001)	(0.005)	(0.005)	(0.001)	(0.001)	(0.005)	(0.005)
Young Firm		-0.108***		-0.148***		-0.108***		-0.148***
		(0.004)		(0.005)		(0.004)		(0.005)
$\theta_{ip} \times$ Young		0.025***		0.076***		0.027***		0.076***
		(0.002)		(0.007)		(0.002)		(0.007)
Constant	0.553***	0.584***	0.646***	0.678***	0.555***	0.585***	0.646***	0.677***
	(0.020)	(0.020)	(0.020)	(0.020)	(0.021)	(0.020)	(0.020)	(0.020)
Observations	1,622,000	1,622,000	1,622,000	1,622,000	1,622,000	1,622,000	1,622,000	1,622,000
Adjusted R^2	0.495	0.497	0.494	0.496	0.495	0.497	0.494	0.496
Controls	Full	Full	Full	Full	Full	Full	Full	Full
θ_{ip} estimation	Norm (jt)	Norm (jt)	Percentile (jt)	Percentile (jt)	Norm (pt)	Norm (pt)	Percentile (pt)	Percentile (pt)

Notes: Columns are based on alternative specification of estimating θ_{ip}. Columns (1)-(2) are based on the first 2 observations for each product-firm pair, (3)-(4) are estimates with 5 year rolling-year basis, (5)-(6) are with 10 year rolling-year basis, (7)-(8) are with 15 year rolling-year basis, (9)-(10) are based on standardization with the industry-year distribution, (11)-(12) are based on the percentile rankings within industry-year, (13)-(14) are based on standardization with the product-year distribution, and (15)-(16) are based on the percentile rankings within product-year. Full-control specifications include the same set of controls as in the baseline regressions. Estimates for product-year and firm fixed effects are suppressed. Observations are unweighted. Standard errors clustered at the SIC5 level are reported in parentheses. * p<0.1, ** p<0.05, *** p<0.01.

Table OA.26: Probability of Product Drop (Alternative θ_{ip} Estimates)

VARIABLES	$Drop_{t:t+5}$	$Drop_{t:t+5}$
θ_{ip}	-0.563***	-0.580***
	(0.016)	(0.017)
Young Firm		-0.317***
		(0.014)
$\theta_{ip} \times$ Young		0.394***
		(0.027)
Constant	0.855***	0.892***
	(0.024)	(0.024)
Observations	1,622,000	1,622,000
Adjusted R^2	0.493	0.495
Controls	Full	Full
θ_{ip} estimation	No controls	No controls

Notes: Results are based on alternative specification of estimating θ_{ip} without firm controls. Estimates for product-year and firm fixed effects are suppressed. Observations are unweighted. Standard errors clustered at the SIC5 level are reported in parentheses. * p<0.1, ** p<0.05, *** p<0.01.

Table OA.27: Probability of Product Drop (θ_{ip}, boostrapped standard errors)

VARIABLES	$Drop_{t:t+5}$	$Drop_{t:t+5}$	$Drop_{t:t+5}$	$Drop_{t:t+5}$	$Drop_{t:t+5}$
θ_{ip}	-1.212***	-1.028***	-1.189***	-1.187***	-1.074***
	(0.094)	(0.119)	(0.095)	(0.095)	(0.120)
log(Emp)			-0.035***	-0.015***	-0.016***
			(0.001)	(0.001)	(0.001)
log(Total Assets)			0.002***	0.023***	0.021***
			(0.000)	(0.001)	(0.001)
# establishments			-0.002***	-0.002***	-0.002***
			(0.000)	(0.000)	(0.000)
# products			0.005***	0.005***	0.005***
			(0.000)	(0.000)	(0.000)
log(CA)			0	0	0.002***
			(0.000)	(0.000)	(0.000)
log(CS)			-0.005***	-0.005***	-0.002***
			(0.000)	(0.000)	(0.000)
TFP			-0.011***	-0.020***	-0.013***
			(0.001)	(0.001)	(0.001)
log(Prod. Emp)				-0.009***	-0.010***
				(0.001)	(0.000)
log(Equipment)				-0.025***	-0.025***
				(0.001)	(0.001)
log(Structure)				0.005***	0.006***
				(0.001)	(0.001)
log(Material)				-0.013***	-0.008***
				(0.001)	(0.001)
log(Energy)				0.001	0.002***
				(0.001)	(0.001)
$\log(PV_{ipt})$		-0.021***			-0.017***
		(0.002)			(0.002)
Tenure		0.041***			0.042***
		(0.000)			(0.000)
Tenure squared		-0.001***			-0.001***
		(0.000)			(0.000)

Notes: Estimates for product-year and firm fixed effects are suppressed. Observations are unweighted. Bootstrapped standard errors are reported in parentheses. * p<0.1, ** p<0.05, *** p<0.01.

Table OA.28: Probability of Product Drop (θ_{ip} × Young, boostrapped standard errors)

VARIABLES	$Drop_{t:t+5}$	$Drop_{t:t+5}$	$Drop_{t:t+5}$	$Drop_{t:t+5}$	$Drop_{t:t+5}$
θ_{ip}	-1.221***	-1.032***	-1.196***	-1.194***	-1.081***
	(0.091)	(0.118)	(0.092)	(0.092)	(0.120)
Young Firm$_{it}$	-0.411***	-0.371***	-0.402***	-0.403***	-0.368***
	(0.055)	(0.042)	(0.056)	(0.056)	(0.043)
θ_{ip} × Young$_{it}$	0.395***	0.514***	0.367***	0.368***	0.497***
	(0.096)	(0.072)	(0.098)	(0.099)	(0.073)
log(Emp)			-0.040***	-0.020***	-0.018***
			(0.001)	(0.001)	(0.001)
log(Total Assets)			0	0.022***	0.020***
			(0.000)	(0.001)	(0.001)
# establishments			-0.002***	-0.002***	-0.001***
			(0.000)	(0.000)	(0.000)
# products			0.005***	0.005***	0.005***
			(0.000)	(0.000)	(0.000)
log(CA)			0	0.001**	0.002***
			(0.000)	(0.000)	(0.000)
log(CS)			-0.004***	-0.004***	-0.002***
			(0.000)	(0.000)	(0.000)
TFP			-0.012***	-0.021***	-0.014***
			(0.001)	(0.001)	(0.001)
log(Prod. Emp)				-0.009***	-0.010***
				(0.001)	(0.000)
log(Equipment)				-0.026***	-0.026***
				(0.001)	(0.001)
log(Structure)				0.005***	0.006***
				(0.001)	(0.001)
log(Material)				-0.014***	-0.008***
				(0.001)	(0.001)
log(Energy)				0	0.002***
				(0.000)	(0.001)
$\log(PV_{ipt})$		-0.021***			-0.017***
		(0.002)			(0.002)
Tenure		0.039***			0.039***
		(0.000)			(0.000)
Tenure squared		-0.001***			-0.001***
		(0.000)			(0.000)

Notes: Estimates for product-year and firm fixed effects are suppressed. Observations are unweighted. Bootstrapped standard errors are reported in parentheses. * p<0.1, ** p<0.05, *** p<0.01.

Table OA.29: Probability of Product Drop (θ_{ip} $\times$ Firm Age)

VARIABLES	$Drop_{t:t+5}$	$Drop_{t:t+5}$
θ_{ip}	-0.816***	-0.628***
	(0.038)	(0.033)
θ_{ip} $\times$ Firm Age$_{it}$	-0.022***	-0.024***
	(0.002)	(0.002)
log(Emp)		-0.015***
		(0.002)
log(Total Assets)		0.021***
		(0.004)
# establishments		-0.002***
		(0.000)
# products		0.005***
		(0.000)
log(CA)		0.001**
		(0.001)
log(CS)		-0.004***
		(0.001)
TFP		-0.013***
		(0.002)
log(Prod. Emp)		-0.010***
		(0.001)
log(Equipment)		-0.025***
		(0.004)
log(Structure)		0.006**
		(0.003)
log(Material)		-0.008***
		(0.001)
log(Energy)		0.002*
		(0.001)
$\log(PV_{ipt})$		-0.018***
		(0.001)
Tenure		0.041***
		(0.001)
Tenure squared		-0.001***
		(0.000)
Constant	1.018***	1.070***
	(0.014)	(0.022)

Notes: Estimates for product-year, firm fixed effects, and the constant are suppressed. Firm Age is measured as the age of the firm in years. Observations are unweighted. * p<0.1, ** p<0.05, *** p<0.01.

Table OA.30: Number of Input Materials Used per Product by Firms

	Including entry and exit			Continuers only		
	all	age 0-5	age 6 +	all	age 0-5	age 6 +
Mean	20.30	19.45	20.43	21.33	21.50	21.31
Stdev.	(20.29)	(19.86)	(20.35)	(21.00)	(21.34)	(20.95)

Notes: Firms in the manufacturing sector from 1982 to 2007.

OA.D.10 Product-Firm Proximity

Up to this point, we have focused on the relationship between product dropping and adding, as well as the relative expertise of added and dropped products. We now turn to examining the proximity between newly added products and a firm's existing product portfolio. The literature finds that firms tend to produce products close to each other in terms of various measures, such as industry classification or the composition of material inputs used (e.g., Bernard et al., 2010; Boehm et al., 2022). Nonetheless, we also observe many instances in which firms expand into entirely new industries.

To examine whether the proximity of added products to a firm's existing product portfolio varies over the firm life-cycle, we use the share of input materials used to produce a newly added product that are also used for the firm's existing products as a proxy for product proximity.[13] Let $MaterialCloseness_{ijt}^{add}$ denote the average of the proximity measure of products added by firm i between t and $t+5$. The mean (standard deviation) of the material-based closeness of added products in the churning case is 0.708 (0.271). Table OA.30 shows the number of input materials used per product by firms across age groups. Focusing on the continuing firms, we find that young firms use slightly more variety of input materials to produce each product, and the variance is also larger compared to mature firms.

We then estimate the following regression model:

$$MaterialCloseness_{ijt}^{add} = \alpha + \beta \mathcal{I}_{ijt}^{young} + X_{ijt}\gamma + \delta_i + \delta_{jt} + \varepsilon_{ijt}. \tag{OA.D.7}$$

Table OA.31 shows that relative to mature firms, young firms tend to add and retain products that are more distant from their existing product portfolio. Results are also consistent with industry-based proximity.

[13] The CMF provides establishment-level data on the list of material inputs used and associated expenditures. By focusing on single-product establishments, we identify the set of material inputs used in the production of each product.

Table OA.31: Material-Based Closeness of Added Products (Young Firm)

	$Mat^{add}_{t:t+5}$	$Mat^{add}_{t:t+5}$	$Mat^{add}_{t:t+5}$
Young Firm$_{it}$	-0.005	-0.003	-0.003
	(0.005)	(0.004)	(0.004)
log(Emp)		-0.012***	-0.003
		(0.002)	(0.003)
log(Total Assets)		-0.004***	-0.001
		(0.001)	(0.004)
# establishments		0.005***	0.005***
		(0.002)	(0.002)
# products		-0.024***	-0.024***
		(0.002)	(0.002)
log(CA)		-0.003***	-0.003***
		(0.001)	(0.001)
log(CS)		-0.007***	-0.007***
		(0.001)	(0.001)
TFP		-0.003	-0.007***
		(0.002)	(0.002)
# materials		0.005***	0.005***
		(0.000)	(0.000)
log(Prod. Emp)			-0.001
			(0.001)
log(Equipment)			-0.005*
			(0.003)
log(Structure)			0.004*
			(0.002)
log(Material)			-0.005***
			(0.001)
log(Energy)			-0.007***
			(0.001)
Constant	0.520***	0.489***	0.521***
	(0.003)	(0.011)	(0.013)
Observations	198,000	198,000	198,000
Fixed effects	i, jt	i, jt	i, jt

Notes: Estimates for industry-year, firm fixed effects, and the constant are suppressed. Robust standard errors clustered at the firm's modal five-digit SIC product level are reported in parentheses. Observations are unweighted, and observation counts are rounded due to the Census Bureau disclosure avoidance procedures. * p<0.1, ** p<0.05, *** p<0.01.

OA.D.11 Alternative Measures of Product-Specific Expertise

An alternative specification replaces time-varying firm and industry controls with product-year and firm-year fixed effects:

$$y_{ipjt} = \theta_{ip} + X_{ipjt}\gamma + \delta_{pt} + \delta_{it} + \varepsilon_{ipjt}, \tag{OA.D.8}$$

where δ_{pt} is a product-year fixed effect, and δ_{it} is a firm-year fixed effect.

δ_{pt} controls for product specific characteristics as well as product-year specific shocks. Again, this is because each product is subject to different production technology or demand structure. Therefore, it is possible that there are several other effects affecting the value of shipment, attributed to product-year specific technology or demand shocks.

In a similar fashion, firms or entrepreneurs have different base sets of available resources or ability to manufacture products (i.e. a financing constraint, customer capital, brand values, and marketing resources, etc.). The value of shipment for each product can also be influenced by these firm-level properties. Furthermore, any types of industry-specific characteristics that can give a substantial impact on the total value of shipment are absorbed by the firm-year fixed effect. Therefore, equation (OA.D.8) could be one alternative of the baseline expertise specification in (3.40).

However, estimation based on (OA.D.8) faces several limitations due to the inclusion of multiple fixed effects. In particular, estimating firm-product and firm-year fixed effects simultaneously leads to the exclusion of a non-negligible portion of the sample, as some products appear in only one firm or some firms produce only a single product in a given year. Therefore, an alternative approach is to replace the firm-year fixed effect δ_{it} with an extensive set of firm-level controls X_{ijt}, which is our baseline specification (3.40).

Alternatively, we could implement an AKM-style specification (Abowd et al., 1999) and construct an alternative expertise measure from the time average of the residuals. These specifications include

$$y_{ipjt} = X_{ipjt}\gamma + \delta_{pt} + \delta_{it} + \varepsilon_{ipjt} \tag{OA.D.9}$$

$$y_{ipjt} = X_{ipjt}\gamma_1 + X_{ijt}\gamma_2 + X_{pt}\gamma_3 + \delta_p + \delta_i + \delta_t + \varepsilon_{ipjt} \tag{OA.D.10}$$

$$y_{ipjt} = X_{ipjt}\gamma_1 + X_{ijt}\gamma_2 + \delta_{pt} + \delta_i + \delta_t + \varepsilon_{ipjt}. \tag{OA.D.11}$$

A critical limitation of this approach is the lack of clarity regarding what the time-varying residuals capture, and whether their average offers a more accurate measure of product-specific expertise

than the baseline specification.

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